1. $\dfrac{dy}{dx} = xy^2 \quad \Rightarrow \quad \dfrac{dy}{y^2} = x\,dx \;\; [y \neq 0] \quad \Rightarrow \quad \displaystyle\int y^{-2}\,dy = \int x\,dx \quad \Rightarrow \quad -y^{-1} = \tfrac{1}{2}x^2 + C \quad \Rightarrow$

$\dfrac{1}{y} = -\tfrac{1}{2}x^2 - C \quad \Rightarrow \quad y = \dfrac{1}{-\frac{1}{2}x^2 - C} = \dfrac{2}{K - x^2}$, where $K = -2C$. $y = 0$ is also a solution.

3. $xy^2y' = x + 1 \quad \Rightarrow \quad y^2\dfrac{dy}{dx} = \dfrac{x+1}{x} \quad \Rightarrow \quad y^2\,dy = \left(1 + \dfrac{1}{x}\right)dx \quad \Rightarrow \quad \displaystyle\int y^2\,dy = \int \left(1 + \dfrac{1}{x}\right)dx \quad \Rightarrow$

$\tfrac{1}{3}y^3 = x + \ln|x| + C \quad \Rightarrow \quad y^3 = 3x + 3\ln|x| + 3C \quad \Rightarrow \quad y = \sqrt[3]{3x + 3\ln|x| + K}$, where $K = 3C$.

5. $(y + \sin y)\,y' = x + x^3 \quad \Rightarrow \quad (y + \sin y)\dfrac{dy}{dx} = x + x^3 \quad \Rightarrow \quad \displaystyle\int (y + \sin y)\,dy = \int (x + x^3)\,dx \quad \Rightarrow$

$\tfrac{1}{2}y^2 - \cos y = \tfrac{1}{2}x^2 + \tfrac{1}{4}x^4 + C$. We cannot solve explicitly for y.

7. $\dfrac{dy}{dt} = \dfrac{t}{ye^{y+t^2}} = \dfrac{t}{ye^y e^{t^2}} \quad \Rightarrow \quad ye^y\,dy = te^{-t^2}\,dt \quad \Rightarrow \quad \displaystyle\int ye^y\,dy = \int te^{-t^2}\,dt \quad \Rightarrow$

$ye^y - e^y$ [by parts] $= -\tfrac{1}{2}e^{-t^2} + C$. The solution is given implicitly by the equation $e^y(y-1) = C - \tfrac{1}{2}e^{-t^2}$.

We cannot solve explicitly for y.

9.
$\dfrac{dp}{dt} = t^2p - p + t^2 - 1 = p(t^2 - 1) + 1(t^2 - 1) = (p+1)(t^2 - 1) \quad \Rightarrow \quad \dfrac{1}{p+1}\,dp = (t^2 - 1)\,dt \quad \Rightarrow$

$\displaystyle\int \frac{1}{p+1}\,dp = \int (t^2 - 1)\,dt \quad \Rightarrow \quad \ln|p+1| = \tfrac{1}{3}t^3 - t + C \quad \Rightarrow \quad |p+1| = e^{t^3/3 - t + C} \quad \Rightarrow \quad p + 1 = \pm e^C e^{t^3/3 - t} \quad \Rightarrow$

$p = Ke^{t^3/3 - t} - 1$, where $K = \pm e^C$. Since $p = -1$ is also a solution, K can equal 0, and hence, K can be any real number.

11.
$\dfrac{dy}{dx} = \dfrac{x}{y} \quad \Rightarrow \quad y\,dy = x\,dx \quad \Rightarrow \quad \int y\,dy = \int x\,dx \quad \Rightarrow \quad \tfrac{1}{2}y^2 = \tfrac{1}{2}x^2 + C. \quad y(0) = -3 \quad \Rightarrow$

$\tfrac{1}{2}(-3)^2 = \tfrac{1}{2}(0)^2 + C \quad \Rightarrow \quad C = \tfrac{9}{2}$, so $\tfrac{1}{2}y^2 = \tfrac{1}{2}x^2 + \tfrac{9}{2} \quad \Rightarrow \quad y^2 = x^2 + 9 \quad \Rightarrow \quad y = -\sqrt{x^2 + 9}$ since $y(0) = -3 < 0$.

13. $\dfrac{du}{dt} = \dfrac{2t + \sec^2 t}{2u}$, $u(0) = -5$. $\quad \int 2u\,du = \int \left(2t + \sec^2 t\right)dt \quad \Rightarrow \quad u^2 = t^2 + \tan t + C$,

where $[u(0)]^2 = 0^2 + \tan 0 + C \quad \Rightarrow \quad C = (-5)^2 = 25$. Therefore, $u^2 = t^2 + \tan t + 25$, so $u = \pm\sqrt{t^2 + \tan t + 25}$.

Since $u(0) = -5$, we must have $u = -\sqrt{t^2 + \tan t + 25}$.

15. $x\ln x = y\left(1+\sqrt{3+y^2}\right)y'$, $y(1)=1$. $\int x\ln x\,dx = \int\left(y+y\sqrt{3+y^2}\right)\,dy \quad\Rightarrow\quad \frac{1}{2}x^2\ln x - \int\frac{1}{2}x\,dx$

[use parts with $u=\ln x$, $dv = x dx$] $= \frac{1}{2}y^2+\frac{1}{3}(3+y^2)^{3/2} \quad\Rightarrow\quad \frac{1}{2}x^2\ln x-\frac{1}{4}x^2+C=\frac{1}{2}y^2+\frac{1}{3}(3+y^2)^{3/2}$.

Now $y(1)=1 \quad\Rightarrow\quad 0-\frac{1}{4}+C=\frac{1}{2}+\frac{1}{3}(4)^{3/2} \quad\Rightarrow\quad C=\frac{1}{2}+\frac{8}{3}+\frac{1}{4}=\frac{41}{12}$, so

$\frac{1}{2}x^2\ln x-\frac{1}{4}x^2+\frac{41}{12}=\frac{1}{2}y^2+\frac{1}{3}(3+y^2)^{3/2}$. We do not solve explicitly for y.

17. $y'\tan x = a+y,\ 0<x<\pi/2 \quad\Rightarrow\quad \dfrac{dy}{dx}=\dfrac{a+y}{\tan x} \quad\Rightarrow\quad \dfrac{dy}{a+y}=\cot x\,dx \quad [a+y\neq 0] \quad\Rightarrow$

$\displaystyle\int\frac{dy}{a+y}=\int\frac{\cos x}{\sin x}\,dx \quad\Rightarrow\quad \ln|a+y|=\ln|\sin x|+C \quad\Rightarrow\quad |a+y|=e^{\ln|\sin x|+C}=e^{\ln|\sin x|}\cdot e^C=e^C\,|\sin x| \quad\Rightarrow$

$a+y=K\sin x$, where $K=\pm e^C$. (In our derivation, K was nonzero, but we can restore the excluded case

$y=-a$ by allowing K to be zero.) $\quad y(\pi/3)=a \quad\Rightarrow\quad a+a=K\sin\left(\dfrac{\pi}{3}\right) \quad\Rightarrow\quad 2a=K\dfrac{\sqrt{3}}{2} \quad\Rightarrow\quad K=\dfrac{4a}{\sqrt{3}}$.

Thus, $a+y=\dfrac{4a}{\sqrt{3}}\sin x$ and so $y=\dfrac{4a}{\sqrt{3}}\sin x-a$.

19. If the slope at the point (x,y) is xy, then we have $\dfrac{dy}{dx}=xy \quad\Rightarrow\quad \dfrac{dy}{y}=x\,dx \quad [y\neq 0] \quad\Rightarrow\quad \displaystyle\int\frac{dy}{y}=\int x\,dx \quad\Rightarrow$

$\ln|y|=\frac{1}{2}x^2+C. \quad y(0)=1 \quad\Rightarrow\quad \ln 1=0+C \quad\Rightarrow\quad C=0$. Thus, $|y|=e^{x^2/2} \quad\Rightarrow\quad y=\pm e^{x^2/2}$, so $y=e^{x^2/2}$

since $y(0)=1>0$. Note that $y=0$ is not a solution because it doesn't satisfy the initial condition $y(0)=1$.

21. $u=x+y \quad\Rightarrow\quad \dfrac{d}{dx}(u)=\dfrac{d}{dx}(x+y) \quad\Rightarrow\quad \dfrac{du}{dx}=1+\dfrac{dy}{dx}$, but $\dfrac{dy}{dx}=x+y=u$, so $\dfrac{du}{dx}=1+u \quad\Rightarrow$

$\dfrac{du}{1+u}=dx \quad [u\neq -1] \quad\Rightarrow\quad \displaystyle\int\frac{du}{1+u}=\int dx \quad\Rightarrow\quad \ln|1+u|=x+C \quad\Rightarrow\quad |1+u|=e^{x+C} \quad\Rightarrow$

$1+u=\pm e^Ce^x \quad\Rightarrow\quad u=\pm e^Ce^x-1 \quad\Rightarrow\quad x+y=\pm e^Ce^x-1 \quad\Rightarrow\quad y=Ke^x-x-1$, where $K=\pm e^C\neq 0$.

If $u=-1$, then $-1=x+y \quad\Rightarrow\quad y=-x-1$, which is just $y=Ke^x-x-1$ with $K=0$. Thus, the general solution is $y=Ke^x-x-1$, where $K\in\mathbb{R}$.