5. $\int_3^\infty \frac{1}{(x-2)^{3/2}}\,dx = \lim_{t\to\infty}\int_3^t (x-2)^{-3/2}\,dx = \lim_{t\to\infty}\left[-2\,(x-2)^{-1/2}\right]_3^t$ $\quad [u = x-2,\, du = dx]$

$= \lim_{t\to\infty}\left(\frac{-2}{\sqrt{t-2}} + \frac{2}{\sqrt{1}}\right) = 0 + 2 = 2.$ Convergent

7. $\int_{-\infty}^0 \frac{1}{3-4x}\,dx = \lim_{t\to-\infty}\int_t^0 \frac{1}{3-4x}\,dx = \lim_{t\to-\infty}\left[-\frac{1}{4}\ln|3-4x|\right]_t^0 = \lim_{t\to-\infty}\left[-\frac{1}{4}\ln 3 + \frac{1}{4}\ln|3-4t|\right] = \infty.$

Divergent

9. $\int_2^\infty e^{-5p}\,dp = \lim_{t\to\infty}\int_2^t e^{-5p}\,dp = \lim_{t\to\infty}\left[-\frac{1}{5}e^{-5p}\right]_2^t = \lim_{t\to\infty}\left(-\frac{1}{5}e^{-5t} + \frac{1}{5}e^{-10}\right) = 0 + \frac{1}{5}e^{-10} = \frac{1}{5}e^{-10}.$ Convergent

11. $\int_0^\infty \frac{x^2}{\sqrt{1+x^3}}\,dx = \lim_{t\to\infty}\int_0^t \frac{x^2}{\sqrt{1+x^3}}\,dx = \lim_{t\to\infty}\left[\frac{2}{3}\sqrt{1+x^3}\right]_0^t = \lim_{t\to\infty}\left(\frac{2}{3}\sqrt{1+t^3} - \frac{2}{3}\right) = \infty.$ Divergent

13. $\int_{-\infty}^\infty xe^{-x^2}\,dx = \int_{-\infty}^0 xe^{-x^2}\,dx + \int_0^\infty xe^{-x^2}\,dx.$

$\int_{-\infty}^0 xe^{-x^2}\,dx = \lim_{t\to-\infty}\left(-\frac{1}{2}\right)\left[e^{-x^2}\right]_t^0 = \lim_{t\to-\infty}\left(-\frac{1}{2}\right)\left(1 - e^{-t^2}\right) = -\frac{1}{2}\cdot 1 = -\frac{1}{2}$, and

$\int_0^\infty xe^{-x^2}\,dx = \lim_{t\to\infty}\left(-\frac{1}{2}\right)\left[e^{-x^2}\right]_0^t = \lim_{t\to\infty}\left(-\frac{1}{2}\right)\left(e^{-t^2} - 1\right) = -\frac{1}{2}\cdot(-1) = \frac{1}{2}.$

Therefore, $\int_{-\infty}^\infty xe^{-x^2}\,dx = -\frac{1}{2} + \frac{1}{2} = 0.$ Convergent

15. $\int_0^\infty \sin^2\alpha\,d\alpha = \lim_{t\to\infty}\int_0^t \frac{1}{2}(1-\cos 2\alpha)\,d\alpha = \lim_{t\to\infty}\left[\frac{1}{2}\left(\alpha - \frac{1}{2}\sin 2\alpha\right)\right]_0^t = \lim_{t\to\infty}\left[\frac{1}{2}\left(t - \frac{1}{2}\sin 2t\right) - 0\right] = \infty.$

Divergent

17. $\int_1^\infty \frac{1}{x^2+x}\,dx = \lim_{t\to\infty}\int_1^t \frac{1}{x(x+1)}\,dx = \lim_{t\to\infty}\int_1^t \left(\frac{1}{x} - \frac{1}{x+1}\right)dx$ $\quad$ [partial fractions]

$= \lim_{t\to\infty}\left[\ln|x| - \ln|x+1|\right]_1^t = \lim_{t\to\infty}\left[\ln\left|\frac{x}{x+1}\right|\right]_1^t = \lim_{t\to\infty}\left(\ln\frac{t}{t+1} - \ln\frac{1}{2}\right) = 0 - \ln\frac{1}{2} = \ln 2.$

Convergent

19. $\int_{-\infty}^0 ze^{2z}\,dz = \lim_{t\to-\infty}\int_t^0 ze^{2z}\,dz = \lim_{t\to-\infty}\left[\frac{1}{2}ze^{2z} - \frac{1}{4}e^{2z}\right]_t^0$ $\quad \left[\begin{array}{c}\text{integration by parts with}\\ u = z,\, dv = e^{2z}\,dz\end{array}\right]$

$= \lim_{t\to-\infty}\left[\left(0 - \frac{1}{4}\right) - \left(\frac{1}{2}te^{2t} - \frac{1}{4}e^{2t}\right)\right] = -\frac{1}{4} - 0 + 0$ [by l'Hospital's Rule] $= -\frac{1}{4}.$ Convergent

21. $\int_1^\infty \frac{\ln x}{x}\,dx = \lim_{t\to\infty}\left[\frac{(\ln x)^2}{2}\right]_1^t$ $\quad \left[\begin{array}{l}\text{by substitution with}\\ u = \ln x,\, du = dx/x\end{array}\right]$ $\quad = \lim_{t\to\infty}\frac{(\ln t)^2}{2} = \infty.$ Divergent

23. $\int_{-\infty}^{\infty} \frac{x^2}{9+x^6}\,dx = \int_{-\infty}^{0} \frac{x^2}{9+x^6}\,dx + \int_{0}^{\infty} \frac{x^2}{9+x^6}\,dx = 2\int_{0}^{\infty} \frac{x^2}{9+x^6}\,dx$ [since the integrand is even].

Now $\int \frac{x^2\,dx}{9+x^6}$ $\left[\begin{matrix} u = x^3 \\ du = 3x^2\,dx \end{matrix}\right]$ $= \int \frac{\frac{1}{3}\,du}{9+u^2}$ $\left[\begin{matrix} u = 3v \\ du = 3\,dv \end{matrix}\right]$ $= \int \frac{\frac{1}{3}(3\,dv)}{9+9v^2} = \frac{1}{9}\int \frac{dv}{1+v^2}$

$$= \frac{1}{9}\tan^{-1} v + C = \frac{1}{9}\tan^{-1}\left(\frac{u}{3}\right) + C = \frac{1}{9}\tan^{-1}\left(\frac{x^3}{3}\right) + C,$$

so $2\int_0^{\infty} \frac{x^2}{9+x^6}\,dx = 2\lim_{t\to\infty}\int_0^t \frac{x^2}{9+x^6}\,dx = 2\lim_{t\to\infty}\left[\frac{1}{9}\tan^{-1}\left(\frac{x^3}{3}\right)\right]_0^t = 2\lim_{t\to\infty}\frac{1}{9}\tan^{-1}\left(\frac{t^3}{3}\right) = \frac{2}{9}\cdot\frac{\pi}{2} = \frac{\pi}{9}.$

Convergent

25. $\int_e^{\infty} \frac{1}{x(\ln x)^3}\,dx = \lim_{t\to\infty}\int_e^t \frac{1}{x(\ln x)^3}\,dx = \lim_{t\to\infty}\int_1^{\ln t} u^{-3}\,du$ $\left[\begin{matrix} u = \ln x, \\ du = dx/x \end{matrix}\right]$ $= \lim_{t\to\infty}\left[-\frac{1}{2u^2}\right]_1^{\ln t}$

$$= \lim_{t\to\infty}\left[-\frac{1}{2(\ln t)^2} + \frac{1}{2}\right] = 0 + \frac{1}{2} = \frac{1}{2}.$$ Convergent

31. $\int_{-2}^{3} \frac{dx}{x^4} = \int_{-2}^{0} \frac{dx}{x^4} + \int_{0}^{3} \frac{dx}{x^4}$, but $\int_{-2}^{0} \frac{dx}{x^4} = \lim_{t\to 0^-}\left[-\frac{x^{-3}}{3}\right]_{-2}^{t} = \lim_{t\to 0^-}\left[-\frac{1}{3t^3} - \frac{1}{24}\right] = \infty.$ Divergent

35. $I = \int_0^3 \frac{dx}{x^2-6x+5} = \int_0^3 \frac{dx}{(x-1)(x-5)} = I_1 + I_2 = \int_0^1 \frac{dx}{(x-1)(x-5)} + \int_1^3 \frac{dx}{(x-1)(x-5)}.$

Now $\frac{1}{(x-1)(x-5)} = \frac{A}{x-1} + \frac{B}{x-5} \quad\Rightarrow\quad 1 = A(x-5) + B(x-1).$

Set $x = 5$ to get $1 = 4B$, so $B = \frac{1}{4}$. Set $x = 1$ to get $1 = -4A$, so $A = -\frac{1}{4}$. Thus

$$I_1 = \lim_{t\to 1^-}\int_0^t \left(\frac{-\frac{1}{4}}{x-1} + \frac{\frac{1}{4}}{x-5}\right)dx = \lim_{t\to 1^-}\left[-\tfrac{1}{4}\ln|x-1| + \tfrac{1}{4}\ln|x-5|\right]_0^t$$
$$= \lim_{t\to 1^-}\left[\left(-\tfrac{1}{4}\ln|t-1| + \tfrac{1}{4}\ln|t-5|\right) - \left(-\tfrac{1}{4}\ln|-1| + \tfrac{1}{4}\ln|-5|\right)\right]$$
$$= \infty, \quad \text{since } \lim_{t\to 1^-}\left(-\tfrac{1}{4}\ln|t-1|\right) = \infty.$$

Since I_1 is divergent, I is divergent.

49.

For $x > 0$, $\frac{x}{x^3+1} < \frac{x}{x^3} = \frac{1}{x^2}$. $\int_1^{\infty} \frac{1}{x^2}\,dx$ is convergent by Equation 2 with $p = 2 > 1$, so $\int_1^{\infty} \frac{x}{x^3+1}\,dx$ is convergent by the Comparison Theorem. $\int_0^1 \frac{x}{x^3+1}\,dx$ is a constant, so $\int_0^{\infty} \frac{x}{x^3+1}\,dx = \int_0^1 \frac{x}{x^3+1}\,dx + \int_1^{\infty} \frac{x}{x^3+1}\,dx$ is also convergent.

51.

For $x > 1$, $f(x) = \dfrac{x+1}{\sqrt{x^4 - x}} > \dfrac{x+1}{\sqrt{x^4}} > \dfrac{x}{x^2} = \dfrac{1}{x}$, so $\int_2^\infty f(x)\,dx$ diverges by comparison with $\int_2^\infty \dfrac{1}{x}\,dx$, which diverges by Equation 2 with $p = 1 \le 1$. Thus, $\int_1^\infty f(x)\,dx = \int_1^2 f(x)\,dx + \int_2^\infty f(x)\,dx$ also diverges.

54.

For $0 < x \le 1$, $\dfrac{\sin^2 x}{\sqrt{x}} \le \dfrac{1}{\sqrt{x}}$. Now

$I = \int_0^\pi \dfrac{1}{\sqrt{x}}\,dx = \lim_{t \to 0^+} \int_t^\pi x^{-1/2}\,dx = \lim_{t \to 0^+} \left[2x^{1/2}\right]_t^\pi = \lim_{t \to 0^+} \left(2\pi - 2\sqrt{t}\right) = 2\pi - 0 = 2\pi$, so I is convergent, and by comparison, $\int_0^\pi \dfrac{\sin^2 x}{\sqrt{x}}\,dx$ is convergent.