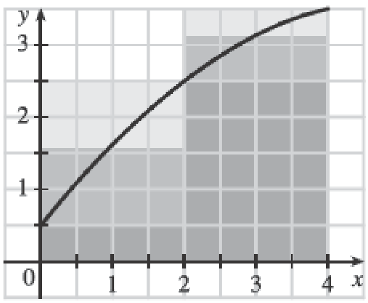
**1.** (a) $\Delta x = (b-a)/n = (4-0)/2 = 2$

$$L_2 = \sum_{i=1}^{2} f(x_{i-1})\,\Delta x = f(x_0)\cdot 2 + f(x_1)\cdot 2 = 2\,[f(0)+f(2)] = 2(0.5+2.5) = 6$$

$$R_2 = \sum_{i=1}^{2} f(x_i)\,\Delta x = f(x_1)\cdot 2 + f(x_2)\cdot 2 = 2\,[f(2)+f(4)] = 2(2.5+3.5) = 12$$

$$M_2 = \sum_{i=1}^{2} f(\overline{x}_i)\Delta x = f(\overline{x}_1)\cdot 2 + f(\overline{x}_2)\cdot 2 = 2\,[f(1)+f(3)] \approx 2(1.6+3.2) = 9.6$$

(b)



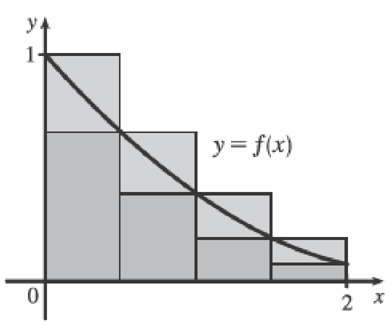

$L_2$ is an underestimate, since the area under the small rectangles is less than the area under the curve, and $R_2$ is an overestimate, since the area under the large rectangles is greater than the area under the curve. It appears that $M_2$ is an overestimate, though it is fairly close to $I$. See the solution to Exercise 47 for a proof of the fact that if $f$ is concave down on $[a, b]$, then the Midpoint Rule is an overestimate of $\int_a^b f(x)\,dx$.

(c) $T_2 = \left(\frac{1}{2}\,\Delta x\right)[f(x_0) + 2f(x_1) + f(x_2)] = \frac{2}{2}[f(0) + 2f(2) + f(4)] = 0.5 + 2(2.5) + 3.5 = 9.$

This approximation is an underestimate, since the graph is concave down. Thus, $T_2 = 9 < I$. See the solution to Exercise 47 for a general proof of this conclusion.

(d) For any $n$, we will have $L_n < T_n < I < M_n < R_n$.

**2.**




The diagram shows that $L_4 > T_4 > \int_0^2 f(x)\,dx > R_4$, and it appears that $M_4$ is a bit less than $\int_0^2 f(x)\,dx$. In fact, for any function that is concave upward, it can be shown that $L_n > T_n > \int_0^2 f(x)\,dx > M_n > R_n$.

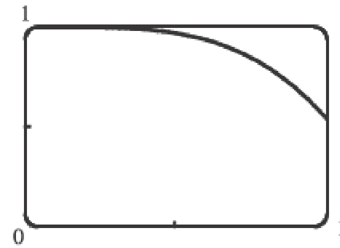
(a) Since $0.9540 > 0.8675 > 0.8632 > 0.7811$, it follows that $L_n = 0.9540$, $T_n = 0.8675$, $M_n = 0.8632$, and $R_n = 0.7811$.

(b) Since $M_n < \int_0^2 f(x)\,dx < T_n$, we have $0.8632 < \int_0^2 f(x)\,dx < 0.8675$.

**3.** $f(x) = \cos(x^2)$, $\Delta x = \frac{1-0}{4} = \frac{1}{4}$

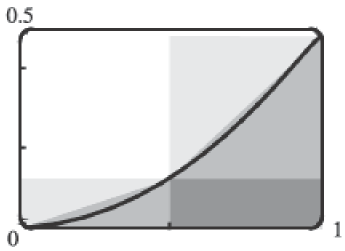
(a) $T_4 = \frac{1}{4 \cdot 2}\left[f(0) + 2f\left(\frac{1}{4}\right) + 2f\left(\frac{2}{4}\right) + 2f\left(\frac{3}{4}\right) + f(1)\right] \approx 0.895759$

(b) $M_4 = \frac{1}{4}\left[f\left(\frac{1}{8}\right) + f\left(\frac{3}{8}\right) + f\left(\frac{5}{8}\right) + f\left(\frac{7}{8}\right)\right] \approx 0.908907$




The graph shows that $f$ is concave down on $[0, 1]$. So $T_4$ is an underestimate and $M_4$ is an overestimate. We can conclude that $0.895759 < \int_0^1 \cos(x^2)\, dx < 0.908907$.

**4.**




(a) Since $f$ is increasing on $[0, 1]$, $L_2$ will underestimate $I$ (since the area of the darkest rectangle is less than the area under the curve), and $R_2$ will overestimate $I$. Since $f$ is concave upward on $[0, 1]$, $M_2$ will underestimate $I$ and $T_2$ will overestimate $I$ (the area under the straight line segments is greater than the area under the curve).

(b) For any $n$, we will have $L_n < M_n < I < T_n < R_n$.

(c) $L_5 = \sum_{i=1}^{5} f(x_{i-1})\ \Delta x = \frac{1}{5}[f(0.0) + f(0.2) + f(0.4) + f(0.6) + f(0.8)] \approx 0.1187$

$R_5 = \sum_{i=1}^{5} f(x_i)\ \Delta x = \frac{1}{5}[f(0.2) + f(0.4) + f(0.6) + f(0.8) + f(1)] \approx 0.2146$

$M_5 = \sum_{i=1}^{5} f(\overline{x}_i)\ \Delta x = \frac{1}{5}[f(0.1) + f(0.3) + f(0.5) + f(0.7) + f(0.9)] \approx 0.1622$

$T_5 = \left(\frac{1}{2}\,\Delta x\right)[f(0) + 2f(0.2) + 2f(0.4) + 2f(0.6) + 2f(0.8) + f(1)] \approx 0.1666$

From the graph, it appears that the Midpoint Rule gives the best approximation. (This is in fact the case, since $I \approx 0.16371405$.)

**5.** (a) $f(x) = \dfrac{x}{1+x^2}$, $\quad \Delta x = \dfrac{b-a}{n} = \dfrac{2-0}{10} = \dfrac{1}{5}$

$M_{10} = \frac{1}{5}\left[f\left(\frac{1}{10}\right) + f\left(\frac{3}{10}\right) + f\left(\frac{5}{10}\right) + \cdots + f\left(\frac{19}{10}\right)\right] \approx 0.806598$

(b) $S_{10} = \frac{1}{5 \cdot 3}\left[f(0) + 4f\left(\frac{1}{5}\right) + 2f\left(\frac{2}{5}\right) + 4f\left(\frac{3}{5}\right) + 2f\left(\frac{4}{5}\right) + \cdots + 4f\left(\frac{9}{5}\right) + f(2)\right] \approx 0.804779$

*Actual:* $I = \displaystyle\int_0^2 \frac{x}{1+x^2}\, dx = \left[\tfrac{1}{2}\ln\left|1+x^2\right|\right]_0^2 \qquad [u = 1 + x^2,\ du = 2x\, dx]$

$= \frac{1}{2}\ln 5 - \frac{1}{2}\ln 1 = \frac{1}{2}\ln 5 \approx 0.804719$

*Errors:* $E_M = \text{actual} - M_{10} = I - M_{10} \approx -0.001879$

$E_S = \text{actual} - S_{10} = I - S_{10} \approx -0.000060$

**6.** (a) $f(x) = x\cos x, \quad \Delta x = \dfrac{b-a}{n} = \dfrac{\pi - 0}{4} = \dfrac{\pi}{4}$

$$M_4 = \tfrac{\pi}{4}\left[f\left(\tfrac{\pi}{8}\right) + f\left(\tfrac{3\pi}{8}\right) + f\left(\tfrac{5\pi}{8}\right) + f\left(\tfrac{7\pi}{8}\right)\right] \approx -1.945744$$

(b) $S_4 = \frac{\pi}{4\cdot 3}\left[f(0) + 4f\left(\frac{\pi}{4}\right) + 2f\left(\frac{2\pi}{4}\right) + 4f\left(\frac{3\pi}{4}\right) + f(\pi)\right] \approx -1.985611$

*Actual:* $I = \int_0^\pi x\cos x\,dx = \Big[x\sin x + \cos x\Big]_0^\pi$ [use parts with $u = x$ and $dv = \cos x\,dx$]

$$= (0 + (-1)) - (0 + 1) = -2$$

*Errors:* $E_M = \text{actual} - M_4 = I - M_4 \approx -0.054256$

$E_S = \text{actual} - S_4 = I - S_4 \approx -0.014389$

**7.**

$f(x) = \sqrt{x^3 - 1}, \Delta x = \dfrac{b-a}{n} = \dfrac{2-1}{10} = \dfrac{1}{10}$

(a) $T_{10} = \frac{1}{10\cdot 2}[f(1) + 2f(1.1) + 2f(1.2) + 2f(1.3) + 2f(1.4) + 2f(1.5)$
$+ 2f(1.6) + 2f(1.7) + 2f(1.8) + 2f(1.9) + f(2)]$
$\approx 1.506361$

(b) $M_{10} = \frac{1}{10}[f(1.05) + f(1.15) + f(1.25) + f(1.35) + f(1.45) + f(1.55) + f(1.65) + f(1.75) + f(1.85) + f(1.95)]$
$\approx 1.518362$

(c) $S_{10} = \frac{1}{10\cdot 3}[f(1) + 4f(1.1) + 2f(1.2) + 4f(1.3) + 2f(1.4)$
$+ 4f(1.5) + 2f(1.6) + 4f(1.7) + 2f(1.8) + 4f(1.9) + f(2)]$
$\approx 1.511519$

**8.** $f(x) = \dfrac{1}{1+x^6}, \Delta x = \dfrac{b-a}{n} = \dfrac{2-0}{8} = \dfrac{1}{4}$

(a) $T_8 = \frac{1}{4\cdot 2}[f(0) + 2f(0.25) + 2f(0.5) + 2f(0.75) + 2f(1) + 2f(1.25) + 2f(1.5) + 2f(1.75) + f(2)] \approx 1.040756$

(b) $M_8 = \frac{1}{4}[f(0.125) + f(0.375) + f(0.625) + f(0.875) + f(1.125) + f(1.375) + f(1.625) + f(1.875)] \approx 1.041109$

(c) $S_8 = \frac{1}{4\cdot 3}[f(0) + 4f(0.25) + 2f(0.5) + 4f(0.75) + 2f(1) + 4f(1.25) + 2f(1.5) + 4f(1.75) + f(2)] \approx 1.042172$

**9.** $f(x) = \dfrac{e^x}{1+x^2}, \Delta x = \dfrac{b-a}{n} = \dfrac{2-0}{10} = \dfrac{1}{5}$

(a) $T_{10} = \frac{1}{5\cdot 2}[f(0) + 2f(0.2) + 2f(0.4) + 2f(0.6) + 2f(0.8) + 2f(1)$
$+ 2f(1.2) + 2f(1.4) + 2f(1.6) + 2f(1.8) + f(2)]$
$\approx 2.660833$

(b) $M_{10} = \frac{1}{5}[f(0.1) + f(0.3) + f(0.5) + f(0.7) + f(0.9) + f(1.1) + f(1.3) + f(1.5) + f(1.7) + f(1.9)]$
$\approx 2.664377$

(c) $S_{10} = \frac{1}{5\cdot 3}[f(0) + 4f(0.2) + 2f(0.4) + 4f(0.6) + 2f(0.8)$
$+ 4f(1) + 2f(1.2) + 4f(1.4) + 2f(1.6) + 4f(1.8) + f(2)]$
$\approx 2.663244$

**10.** $f(x) = \sqrt[3]{1+\cos x}$, $\Delta x = \frac{\pi/2 - 0}{4} = \frac{\pi}{8}$

(a) $T_4 = \frac{\pi}{8 \cdot 2}\left[f(0) + 2f\left(\frac{\pi}{8}\right) + 2f\left(\frac{2\pi}{8}\right) + 2f\left(\frac{3\pi}{8}\right) + f\left(\frac{\pi}{2}\right)\right] \approx 1.838967$

(b) $M_4 = \frac{\pi}{8}\left[f\left(\frac{\pi}{16}\right) + f\left(\frac{3\pi}{16}\right) + f\left(\frac{5\pi}{16}\right) + f\left(\frac{7\pi}{16}\right)\right] \approx 1.845390$

(c) $S_4 = \frac{\pi}{8 \cdot 3}\left[f(0) + 4f\left(\frac{\pi}{8}\right) + 2f\left(\frac{2\pi}{8}\right) + 4f\left(\frac{3\pi}{8}\right) + f\left(\frac{\pi}{2}\right)\right] \approx 1.843245$

**11.** $f(x) = \sqrt{\ln x}$, $\Delta x = \frac{4-1}{6} = \frac{1}{2}$

(a) $T_6 = \frac{1}{2 \cdot 2}[f(1) + 2f(1.5) + 2f(2) + 2f(2.5) + 2f(3) + 2f(3.5) + f(4)] \approx 2.591334$

(b) $M_6 = \frac{1}{2}[f(1.25) + f(1.75) + f(2.25) + f(2.75) + f(3.25) + f(3.75)] \approx 2.681046$

(c) $S_6 = \frac{1}{2 \cdot 3}[f(1) + 4f(1.5) + 2f(2) + 4f(2.5) + 2f(3) + 4f(3.5) + f(4)] \approx 2.631976$

**12.** $f(x) = \sin(x^3)$, $\Delta x = \frac{1-0}{10} = \frac{1}{10}$

(a) $$T_{10} = \frac{1}{10 \cdot 2}[f(0) + 2f(0.1) + 2f(0.2) + 2f(0.3) + 2f(0.4) + 2f(0.5) + 2f(0.6) + 2f(0.7) + 2f(0.8) + 2f(0.9) + f(1)] \approx 0.235205$$

(b) $$M_{10} = \frac{1}{10}[f(0.05) + f(0.15) + f(0.25) + f(0.35) + f(0.45) + f(0.55) + f(0.65) + f(0.75) + f(0.85) + f(0.95)] \approx 0.233162$$

(c) $$S_{10} = \frac{1}{10 \cdot 3}[f(0) + 4f(0.1) + 2f(0.2) + 4f(0.3) + 2f(0.4) + 4f(0.5) + 2f(0.6) + 4f(0.7) + 2f(0.8) + 4f(0.9) + f(1)] \approx 0.233810$$

**13.** $f(t) = e^{\sqrt{t}}\sin t$, $\Delta t = \frac{4-0}{8} = \frac{1}{2}$

(a) $T_8 = \frac{1}{2 \cdot 2}\left[f(0) + 2f\left(\frac{1}{2}\right) + 2f(1) + 2f\left(\frac{3}{2}\right) + 2f(2) + 2f\left(\frac{5}{2}\right) + 2f(3) + 2f\left(\frac{7}{2}\right) + f(4)\right] \approx 4.513618$

(b) $M_8 = \frac{1}{2}\left[f\left(\frac{1}{4}\right) + f\left(\frac{3}{4}\right) + f\left(\frac{5}{4}\right) + f\left(\frac{7}{4}\right) + f\left(\frac{9}{4}\right) + f\left(\frac{11}{4}\right) + f\left(\frac{13}{4}\right) + f\left(\frac{15}{4}\right)\right] \approx 4.748256$

(c) $S_8 = \frac{1}{2 \cdot 3}\left[f(0) + 4f\left(\frac{1}{2}\right) + 2f(1) + 4f\left(\frac{3}{2}\right) + 2f(2) + 4f\left(\frac{5}{2}\right) + 2f(3) + 4f\left(\frac{7}{2}\right) + f(4)\right] \approx 4.675111$

**14.** $f(z) = \sqrt{z}e^{-z}$, $\Delta z = \frac{1-0}{10} = \frac{1}{10}$

(a) $T_{10} = \frac{1}{10 \cdot 2}\{f(0) + 2[f(0.1) + f(0.2) + \cdots + f(0.9)] + f(1)\} \approx 0.372299$

(b) $M_{10} = \frac{1}{10}[f(0.05) + f(0.15) + f(0.25) + \cdots + f(0.95)] \approx 0.380894$

(c) $$S_{10} = \frac{1}{10 \cdot 3}[f(0) + 4f(0.1) + 2f(0.2) + 4f(0.3) + 2f(0.4) + 4f(0.5) + 2f(0.6) + 4f(0.7) + 2f(0.8) + 4f(0.9) + f(1)] \approx 0.376330$$

**15.** $f(x) = \dfrac{\cos x}{x}$, $\Delta x = \frac{5-1}{8} = \frac{1}{2}$

(a) $T_8 = \frac{1}{2 \cdot 2}\left[f(1) + 2f\left(\frac{3}{2}\right) + 2f(2) + \cdots + 2f(4) + 2f\left(\frac{9}{2}\right) + f(5)\right] \approx -0.495333$

(b) $M_8 = \frac{1}{2}\left[f\left(\frac{5}{4}\right) + f\left(\frac{7}{4}\right) + f\left(\frac{9}{4}\right) + f\left(\frac{11}{4}\right) + f\left(\frac{13}{4}\right) + f\left(\frac{15}{4}\right) + f\left(\frac{17}{4}\right) + f\left(\frac{19}{4}\right)\right] \approx -0.543321$

(c) $S_8 = \frac{1}{2 \cdot 3}\left[f(1) + 4f\left(\frac{3}{2}\right) + 2f(2) + 4f\left(\frac{5}{2}\right) + 2f(3) + 4f\left(\frac{7}{2}\right) + 2f(4) + 4f\left(\frac{9}{2}\right) + f(5)\right] \approx -0.526123$

**16.** $f(x) = \ln(x^3 + 2)$, $\Delta x = \frac{6-4}{10} = \frac{1}{5}$

(a) $T_{10} = \frac{1}{5 \cdot 2}\left[f(4) + 2f(4.2) + 2f(4.4) + \cdots + 2f(5.6) + 2f(5.8) + f(6)\right] \approx 9.649753$

(b) $M_{10} = \frac{1}{5}[f(4.1) + f(4.3) + \cdots + f(5.7) + f(5.9)] \approx 9.650912$

(c) $S_{10} = \frac{1}{5 \cdot 3}[f(4) + 4f(4.2) + 2f(4.4) + 4f(4.6) + 2f(4.8) + 4f(5) + 2f(5.2)$
$+ 4f(5.4) + 2f(5.6) + 4f(5.8) + f(6)]$
$\approx 9.650526$

**17.** $f(x) = e^{e^x}$, $\Delta x = \frac{1-(-1)}{10} = \frac{1}{5}$

(a) $T_{10} = \frac{1}{5 \cdot 2}[f(-1) + 2f(-0.8) + 2f(-0.6) + 2f(-0.4) + 2f(-0.2) + 2f(0)$
$+ 2f(0.2) + 2f(0.4) + 2f(0.6) + 2f(0.8) + f(1)]$
$\approx 8.363853$

(b) $M_{10} = \frac{1}{5}[f(-0.9) + f(-0.7) + f(-0.5) + f(-0.3) + f(-0.1) + f(0.1) + f(0.3) + f(0.5) + f(0.7) + f(0.9)]$
$\approx 8.163298$

(c) $S_{10} = \frac{1}{5 \cdot 3}[f(-1) + 4f(-0.8) + 2f(-0.6) + 4f(-0.4) + 2f(-0.2)$
$+ 4f(0) + 2f(0.2) + 4f(0.4) + 2f(0.6) + 4f(0.8) + f(1)]$
$\approx 8.235114$

**18.** $f(x) = \cos\sqrt{x}$, $\Delta x = \frac{4-0}{10} = \frac{2}{5} = 0.4$

(a) $T_{10} = \frac{2}{5 \cdot 2}\left[f(0) + 2f(0.4) + 2f(0.8) + \cdots + 2f(3.2) + 2f(3.6) + f(4)\right] \approx 0.808532$

(b) $M_{10} = \frac{2}{5}[f(0.2) + f(0.6) + f(1) + \cdots + f(3.4) + f(3.8)] \approx 0.803078$

(c) $S_{10} = \frac{2}{5 \cdot 3}[f(0) + 4f(0.4) + 2f(0.8) + 4f(1.2) + 2f(1.6) + 4f(2) + 2f(2.4)$
$+ 4f(2.8) + 2f(3.2) + 4f(3.6) + f(4)]$

$\approx 0.804896$

**19.** $f(x) = \cos(x^2)$, $\Delta x = \frac{1-0}{8} = \frac{1}{8}$

(a) $T_8 = \frac{1}{8 \cdot 2}\{f(0) + 2[f(\frac{1}{8}) + f(\frac{2}{8}) + \cdots + f(\frac{7}{8})] + f(1)\} \approx 0.902333$

$M_8 = \frac{1}{8}[f(\frac{1}{16}) + f(\frac{3}{16}) + f(\frac{5}{16}) + \cdots + f(\frac{15}{16})] = 0.905620$

(b) $f(x) = \cos(x^2)$, $f'(x) = -2x\sin(x^2)$, $f''(x) = -2\sin(x^2) - 4x^2\cos(x^2)$. For $0 \le x \le 1$, sin and cos are positive, so $|f''(x)| = 2\sin(x^2) + 4x^2\cos(x^2) \le 2 \cdot 1 + 4 \cdot 1 \cdot 1 = 6$ since $\sin(x^2) \le 1$ and $\cos(x^2) \le 1$ for all $x$, and $x^2 \le 1$ for $0 \le x \le 1$. So for $n = 8$, we take $K = 6$, $a = 0$, and $b = 1$ in Theorem 3, to get $|E_T| \le 6 \cdot 1^3/(12 \cdot 8^2) = \frac{1}{128} = 0.0078125$ and $|E_M| \le \frac{1}{256} = 0.00390625$. [A better estimate is obtained by noting from a graph of $f''$ that $|f''(x)| \le 4$ for $0 \le x \le 1$.]

(c) Take $K = 6$ [as in part (b)] in Theorem 3. $|E_T| \le \dfrac{K(b-a)^3}{12n^2} \le 0.0001 \quad \Leftrightarrow \quad \dfrac{6(1-0)^3}{12n^2} \le 10^{-4} \quad \Leftrightarrow$

$\dfrac{1}{2n^2} \le \dfrac{1}{10^4} \quad \Leftrightarrow \quad 2n^2 \ge 10^4 \quad \Leftrightarrow \quad n^2 \ge 5000 \quad \Leftrightarrow \quad n \ge 71$. Take $n = 71$ for $T_n$. For $E_M$, again take $K = 6$ in Theorem 3 to get $|E_M| \le 10^{-4} \quad \Leftrightarrow \quad 4n^2 \ge 10^4 \quad \Leftrightarrow \quad n^2 \ge 2500 \quad \Leftrightarrow \quad n \ge 50$. Take $n = 50$ for $M_n$.

**20.** $f(x) = e^{1/x}$, $\Delta x = \frac{2-1}{10} = \frac{1}{10}$

(a) $T_{10} = \frac{1}{10 \cdot 2}[f(1) + 2f(1.1) + 2f(1.2) + \cdots + 2f(1.9) + f(2)] \approx 2.021976$

$M_{10} = \frac{1}{10}[f(1.05) + f(1.15) + f(1.25) + \cdots + f(1.95)] \approx 2.019102$

(b) $f(x) = e^{1/x}$, $f'(x) = -\dfrac{1}{x^2}e^{1/x}$, $f''(x) = \dfrac{2x+1}{x^4}e^{1/x}$. Now $f''$ is decreasing on $[1, 2]$, so let $x = 1$ to take $K = 3e$.

$|E_T| \le \dfrac{3e(2-1)^3}{12(10)^2} = \dfrac{e}{400} \approx 0.006796$. $\quad |E_M| \le \dfrac{|E_T|}{2} = \dfrac{e}{800} \approx 0.003398$.

(c) Take $K = 3e$ [as in part (b)] in Theorem 3. $|E_T| \le \dfrac{K(b-a)^3}{12n^2} \le 0.0001 \quad \Leftrightarrow \quad \dfrac{3e(2-1)^3}{12n^2} \le 10^{-4} \quad \Leftrightarrow$

$\dfrac{e}{4n^2} \le \dfrac{1}{10^4} \quad \Leftrightarrow \quad n^2 \ge \dfrac{10^4 e}{4} \quad \Leftrightarrow \quad n \ge 83$. Take $n = 83$ for $T_n$. For $E_M$, again take $K = 3e$ in Theorem 3 to get $|E_M| \le 10^{-4} \quad \Leftrightarrow \quad n^2 \ge \dfrac{10^4 e}{8} \quad \Leftrightarrow \quad n \ge 59$. Take $n = 59$ for $M_n$.

**21.** $f(x) = \sin x$, $\Delta x = \frac{\pi - 0}{10} = \frac{\pi}{10}$

(a) $T_{10} = \frac{\pi}{10 \cdot 2}\left[f(0) + 2f\left(\frac{\pi}{10}\right) + 2f\left(\frac{2\pi}{10}\right) + \cdots + 2f\left(\frac{9\pi}{10}\right) + f(\pi)\right] \approx 1.983524$

$M_{10} = \frac{\pi}{10}\left[f\left(\frac{\pi}{20}\right) + f\left(\frac{3\pi}{20}\right) + f\left(\frac{5\pi}{20}\right) + \cdots + f\left(\frac{19\pi}{20}\right)\right] \approx 2.008248$

$S_{10} = \frac{\pi}{10 \cdot 3}\left[f(0) + 4f\left(\frac{\pi}{10}\right) + 2f\left(\frac{2\pi}{10}\right) + 4f\left(\frac{3\pi}{10}\right) + \cdots + 4f\left(\frac{9\pi}{10}\right) + f(\pi)\right] \approx 2.000110$

Since $I = \int_0^\pi \sin x \, dx = \left[-\cos x\right]_0^\pi = 1 - (-1) = 2$, $E_T = I - T_{10} \approx 0.016476$, $E_M = I - M_{10} \approx -0.008248$, and $E_S = I - S_{10} \approx -0.000110$.

(b) $f(x) = \sin x \quad \Rightarrow \quad \left|f^{(n)}(x)\right| \le 1$, so take $K = 1$ for all error estimates.

$|E_T| \le \dfrac{K(b-a)^3}{12n^2} = \dfrac{1(\pi - 0)^3}{12(10)^2} = \dfrac{\pi^3}{1200} \approx 0.025839. \quad |E_M| \le \dfrac{|E_T|}{2} = \dfrac{\pi^3}{2400} \approx 0.012919.$

$|E_S| \le \dfrac{K(b-a)^5}{180n^4} = \dfrac{1(\pi - 0)^5}{180(10)^4} = \dfrac{\pi^5}{1{,}800{,}000} \approx 0.000170.$

The actual error is about 64% of the error estimate in all three cases.

(c) $|E_T| \le 0.00001 \quad \Leftrightarrow \quad \dfrac{\pi^3}{12n^2} \le \dfrac{1}{10^5} \quad \Leftrightarrow \quad n^2 \ge \dfrac{10^5\pi^3}{12} \quad \Rightarrow \quad n \ge 508.3$. Take $n = 509$ for $T_n$.

$|E_M| \le 0.00001 \quad \Leftrightarrow \quad \dfrac{\pi^3}{24n^2} \le \dfrac{1}{10^5} \quad \Leftrightarrow \quad n^2 \ge \dfrac{10^5\pi^3}{24} \quad \Rightarrow \quad n \ge 359.4$. Take $n = 360$ for $M_n$.

$|E_S| \le 0.00001 \quad \Leftrightarrow \quad \dfrac{\pi^5}{180n^4} \le \dfrac{1}{10^5} \quad \Leftrightarrow \quad n^4 \ge \dfrac{10^5\pi^5}{180} \quad \Rightarrow \quad n \ge 20.3.$

Take $n = 22$ for $S_n$ (since $n$ must be even).

**22.** From Example 7(b), we take $K = 76e$ to get $|E_S| \le \dfrac{76e(1)^5}{180n^4} \le 0.00001 \quad \Rightarrow \quad n^4 \ge \dfrac{76e}{180(0.00001)} \quad \Rightarrow \quad n \ge 18.4$.

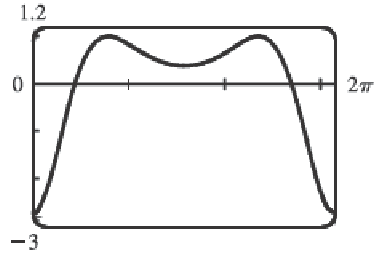
Take $n = 20$ (since $n$ must be even).

**23.**

(a) Using a CAS, we differentiate $f(x) = e^{\cos x}$ twice, and find that

$f''(x) = e^{\cos x}(\sin^2 x - \cos x)$. From the graph, we see that the maximum value of $|f''(x)|$ occurs at the endpoints of the interval $[0, 2\pi]$.

Since $f''(0) = -e$, we can use $K = e$ or $K = 2.8$.




(b) A CAS gives $M_{10} \approx 7.954926518$. (In Maple, use `Student[Calculus1][RiemannSum]` or `Student[Calculus1][ApproximateInt]`.)

(c) Using Theorem 3 for the Midpoint Rule, with $K = e$, we get $|E_M| \le \dfrac{e(2\pi - 0)^3}{24 \cdot 10^2} \approx 0.280945995$.

With $K = 2.8$, we get $|E_M| \le \dfrac{2.8(2\pi - 0)^3}{24 \cdot 10^2} = 0.289391916$.

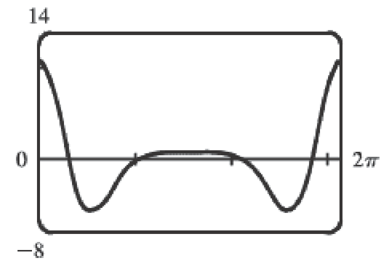
(d) A CAS gives $I \approx 7.954926521$.

(e) The actual error is only about $3 \times 10^{-9}$, much less than the estimate in part (c).

(f) We use the CAS to differentiate twice more, and then graph

$f^{(4)}(x) = e^{\cos x}(\sin^4 x - 6\sin^2 x \cos x + 3 - 7\sin^2 x + \cos x)$.

From the graph, we see that the maximum value of $\left|f^{(4)}(x)\right|$ occurs at the endpoints of the interval $[0, 2\pi]$. Since $f^{(4)}(0) = 4e$, we can use $K = 4e$ or $K = 10.9$.




(g) A CAS gives $S_{10} \approx 7.953789422$. (In Maple, use `Student[Calculus1][ApproximateInt]`.)

(h) Using Theorem 4 with $K = 4e$, we get $|E_S| \le \dfrac{4e(2\pi - 0)^5}{180 \cdot 10^4} \approx 0.059153618$.

With $K = 10.9$, we get $|E_S| \le \dfrac{10.9(2\pi - 0)^5}{180 \cdot 10^4} \approx 0.059299814$.

(i) The actual error is about $7.954926521 - 7.953789422 \approx 0.00114$. This is quite a bit smaller than the estimate in part (h), though the difference is not nearly as great as it was in the case of the Midpoint Rule.

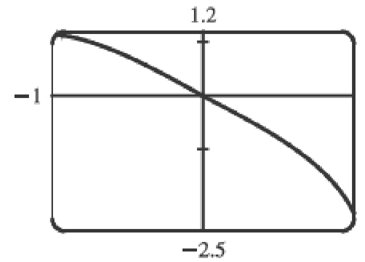
(j) To ensure that $|E_S| \le 0.0001$, we use Theorem 4: $|E_S| \le \dfrac{4e(2\pi)^5}{180 \cdot n^4} \le 0.0001 \quad \Rightarrow \quad \dfrac{4e(2\pi)^5}{180 \cdot 0.0001} \le n^4 \quad \Rightarrow$

$n^4 \ge 5{,}915{,}362 \quad \Leftrightarrow \quad n \ge 49.3$. So we must take $n \ge 50$ to ensure that $|I - S_n| \le 0.0001$.

($K = 10.9$ leads to the same value of $n$.)

**24.** (a) Using the CAS, we differentiate $f(x) = \sqrt{4 - x^3}$ twice, and find that $f''(x) = -\dfrac{9x^4}{4(4 - x^3)^{3/2}} - \dfrac{3x}{(4 - x^3)^{1/2}}$.

From the graph, we see that $|f''(x)| < 2.2$ on $[-1, 1]$.




(b) A CAS gives $M_{10} \approx 3.995804152$. (In Maple, use `Student[Calculus1][RiemannSum]` or `Student[Calculus1][ApproximateInt]`.)

(c) Using Theorem 3 for the Midpoint Rule, with $K = 2.2$, we get $|E_M| \leq \dfrac{2.2\,[1 - (-1)]^3}{24 \cdot 10^2} \approx 0.00733$.

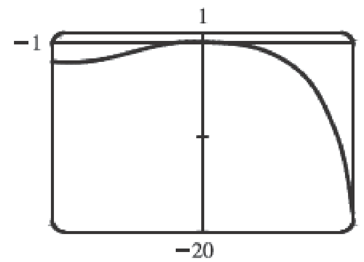
(d) A CAS gives $I \approx 3.995487677$.

(e) The actual error is about $-0.0003165$, much less than the estimate in part (c).

(f) We use the CAS to differentiate twice more, and then graph

$$f^{(4)}(x) = \frac{9}{16}\frac{x^2(x^6 - 224x^3 - 1280)}{(4 - x^3)^{7/2}}.$$

From the graph, we see that $\left|f^{(4)}(x)\right| < 18.1$ on $[-1, 1]$.




(g) A CAS gives $S_{10} \approx 3.995449790$. (In Maple, use `Student[Calculus1][ApproximateInt]`.)

(h) Using Theorem 4 with $K = 18.1$, we get $|E_S| \leq \dfrac{18.1\,[1 - (-1)]^5}{180 \cdot 10^4} \approx 0.000322$.

(i) The actual error is about $3.995487677 - 3.995449790 \approx 0.0000379$. This is quite a bit smaller than the estimate in part (h).

(j) To ensure that $|E_S| \leq 0.0001$, we use Theorem 4: $|E_S| \leq \dfrac{18.1(2)^5}{180 \cdot n^4} \leq 0.0001 \quad \Rightarrow \quad \dfrac{18.1(2)^5}{180 \cdot 0.0001} \leq n^4 \quad \Rightarrow$ $n^4 \geq 32{,}178 \quad \Rightarrow n \geq 13.4$. So we must take $n \geq 14$ to ensure that $|I - S_n| \leq 0.0001$.

**25.** $I = \int_0^1 xe^x dx = [(x-1)e^x]_0^1$ [parts or Formula 96] $= 0 - (-1) = 1$, $f(x) = xe^x$, $\Delta x = 1/n$

$n = 5$: $L_5 = \frac{1}{5}[f(0) + f(0.2) + f(0.4) + f(0.6) + f(0.8)] \approx 0.742943$

$R_5 = \frac{1}{5}[f(0.2) + f(0.4) + f(0.6) + f(0.8) + f(1)] \approx 1.286599$

$T_5 = \frac{1}{5 \cdot 2}[f(0) + 2f(0.2) + 2f(0.4) + 2f(0.6) + 2f(0.8) + f(1)] \approx 1.014771$

$M_5 = \frac{1}{5}[f(0.1) + f(0.3) + f(0.5) + f(0.7) + f(0.9)] \approx 0.992621$

$E_L = I - L_5 \approx 1 - 0.742943 = 0.257057$

$E_R \approx 1 - 1.286599 = -0.286599$

$E_T \approx 1 - 1.014771 = -0.014771$

$E_M \approx 1 - 0.992621 = 0.007379$

$n = 10$: $L_{10} = \frac{1}{10}[f(0) + f(0.1) + f(0.2) + \cdots + f(0.9)] \approx 0.867782$

$R_{10} = \frac{1}{10}[f(0.1) + f(0.2) + \cdots + f(0.9) + f(1)] \approx 1.139610$

$T_{10} = \frac{1}{10 \cdot 2}\{f(0) + 2[f(0.1) + f(0.2) + \cdots + f(0.9)] + f(1)\} \approx 1.003696$

$M_{10} = \frac{1}{10}[f(0.05) + f(0.15) + \cdots + f(0.85) + f(0.95)] \approx 0.998152$

$E_L = I - L_{10} \approx 1 - 0.867782 = 0.132218$

$E_R \approx 1 - 1.139610 = -0.139610$

$E_T \approx 1 - 1.003696 = -0.003696$

$E_M \approx 1 - 0.998152 = 0.001848$

$n = 20$: $L_{20} = \frac{1}{20}[f(0) + f(0.05) + f(0.10) + \cdots + f(0.95)] \approx 0.932967$

$R_{20} = \frac{1}{20}[f(0.05) + f(0.10) + \cdots + f(0.95) + f(1)] \approx 1.068881$

$T_{20} = \frac{1}{20 \cdot 2}\{f(0) + 2[f(0.05) + f(0.10) + \cdots + f(0.95)] + f(1)\} \approx 1.000924$

$M_{20} = \frac{1}{20}[f(0.025) + f(0.075) + f(0.125) + \cdots + f(0.975)] \approx 0.999538$

$E_L = I - L_{20} \approx 1 - 0.932967 = 0.067033$

$E_R \approx 1 - 1.068881 = -0.068881$

$E_T \approx 1 - 1.000924 = -0.000924$

$E_M \approx 1 - 0.999538 = 0.000462$

| $n$ | $L_n$ | $R_n$ | $T_n$ | $M_n$ |
|---|---|---|---|---|
| 5 | 0.742943 | 1.286599 | 1.014771 | 0.992621 |
| 10 | 0.867782 | 1.139610 | 1.003696 | 0.998152 |
| 20 | 0.932967 | 1.068881 | 1.000924 | 0.999538 |

| $n$ | $E_L$ | $E_R$ | $E_T$ | $E_M$ |
|---|---|---|---|---|
| 5 | 0.257057 | −0.286599 | −0.014771 | 0.007379 |
| 10 | 0.132218 | −0.139610 | −0.003696 | 0.001848 |
| 20 | 0.067033 | −0.068881 | −0.000924 | 0.000462 |

*Observations:*

1. $E_L$ and $E_R$ are always opposite in sign, as are $E_T$ and $E_M$.
2. As $n$ is doubled, $E_L$ and $E_R$ are decreased by about a factor of 2, and $E_T$ and $E_M$ are decreased by a factor of about 4.
3. The Midpoint approximation is about twice as accurate as the Trapezoidal approximation.
4. All the approximations become more accurate as the value of $n$ increases.
5. The Midpoint and Trapezoidal approximations are much more accurate than the endpoint approximations.

**26.** $I = \int_1^2 \frac{1}{x^2}\,dx = \left[-\frac{1}{x}\right]_1^2 = -\frac{1}{2} - (-1) = \frac{1}{2}, f(x) = \frac{1}{x^2}, \Delta x = \frac{1}{n}$

$n = 5$:

$L_5 = \frac{1}{5}[f(1) + f(1.2) + f(1.4) + f(1.6) + f(1.8)] \approx 0.580783$

$R_5 = \frac{1}{5}[f(1.2) + f(1.4) + f(1.6) + f(1.8) + f(2)] \approx 0.430783$

$T_5 = \frac{1}{5 \cdot 2}[f(1) + 2f(1.2) + 2f(1.4) + 2f(1.6) + 2f(1.8) + f(2)] \approx 0.505783$

$M_5 = \frac{1}{5}[f(1.1) + f(1.3) + f(1.5) + f(1.7) + f(1.9)] \approx 0.497127$

$E_L = I - L_5 \approx \frac{1}{2} - 0.580783 = -0.080783$

$E_R \approx \frac{1}{2} - 0.430783 = 0.069217$

$E_T \approx \frac{1}{2} - 0.505783 = -0.005783$

$E_M \approx \frac{1}{2} - 0.497127 = 0.002873$

$n = 10$:

$L_{10} = \frac{1}{10}[f(1) + f(1.1) + f(1.2) + \cdots + f(1.9)] \approx 0.538955$

$R_{10} = \frac{1}{10}[f(1.1) + f(1.2) + \cdots + f(1.9) + f(2)] \approx 0.463955$

$T_{10} = \frac{1}{10 \cdot 2}\{f(1) + 2[f(1.1) + f(1.2) + \cdots + f(1.9)] + f(2)\} \approx 0.501455$

$M_{10} = \frac{1}{10}[f(1.05) + f(1.15) + \cdots + f(1.85) + f(1.95)] \approx 0.499274$

$E_L = I - L_{10} \approx \frac{1}{2} - 0.538955 = -0.038955$

$E_R \approx \frac{1}{2} - 0.463955 = 0.036049$

$E_T \approx \frac{1}{2} - 0.501455 = -0.001455$

$E_M \approx \frac{1}{2} - 0.499274 = 0.000726$

$n = 20$:

$L_{20} = \frac{1}{20}[f(1) + f(1.05) + f(1.10) + \cdots + f(1.95)] \approx 0.519114$

$R_{20} = \frac{1}{20}[f(1.05) + f(1.10) + \cdots + f(1.95) + f(2)] \approx 0.481614$

$T_{20} = \frac{1}{20 \cdot 2}\{f(1) + 2[f(1.05) + f(1.10) + \cdots + f(1.95)] + f(2)\} \approx 0.500364$

$M_{20} = \frac{1}{20}[f(1.025) + f(1.075) + f(1.125) + \cdots + f(1.975)] \approx 0.499818$

$E_L = I - L_{20} \approx \frac{1}{2} - 0.519114 = -0.019114$

$E_R \approx \frac{1}{2} - 0.481614 = 0.018386$

$E_T \approx \frac{1}{2} - 0.500364 = -0.000364$

$E_M \approx \frac{1}{2} - 0.499818 = 0.000182$

| $n$ | $L_n$ | $R_n$ | $T_n$ | $M_n$ |
|---|---|---|---|---|
| 5 | 0.580783 | 0.430783 | 0.505783 | 0.497127 |
| 10 | 0.538955 | 0.463955 | 0.501455 | 0.499274 |
| 20 | 0.519114 | 0.481614 | 0.500364 | 0.499818 |

| $n$ | $E_L$ | $E_R$ | $E_T$ | $E_M$ |
|---|---|---|---|---|
| 5 | −0.080783 | 0.069217 | −0.005783 | 0.002873 |
| 10 | −0.038955 | 0.036049 | −0.001455 | 0.000726 |
| 20 | −0.019114 | 0.018386 | −0.000364 | 0.000182 |

*Observations:*

1. $E_L$ and $E_R$ are always opposite in sign, as are $E_T$ and $E_M$.
2. As $n$ is doubled, $E_L$ and $E_R$ are decreased by about a factor of 2, and $E_T$ and $E_M$ are decreased by a factor of about 4.
3. The Midpoint approximation is about twice as accurate as the Trapezoidal approximation.
4. All the approximations become more accurate as the value of $n$ increases.
5. The Midpoint and Trapezoidal approximations are much more accurate than the endpoint approximations.

**27.**

$I = \int_0^2 x^4\,dx = \left[\frac{1}{5}x^5\right]_0^2 = \frac{32}{5} - 0 = 6.4$, $f(x) = x^4$, $\Delta x = \frac{2-0}{n} = \frac{2}{n}$

$n = 6$: $T_6 = \frac{2}{6\cdot 2}\left\{f(0) + 2\left[f\left(\frac{1}{3}\right) + f\left(\frac{2}{3}\right) + f\left(\frac{3}{3}\right) + f\left(\frac{4}{3}\right) + f\left(\frac{5}{3}\right)\right] + f(2)\right\} \approx 6.695473$

$M_6 = \frac{2}{6}\left[f\left(\frac{1}{6}\right) + f\left(\frac{3}{6}\right) + f\left(\frac{5}{6}\right) + f\left(\frac{7}{6}\right) + f\left(\frac{9}{6}\right) + f\left(\frac{11}{6}\right)\right] \approx 6.252572$

$S_6 = \frac{2}{6\cdot 3}\left[f(0) + 4f\left(\frac{1}{3}\right) + 2f\left(\frac{2}{3}\right) + 4f\left(\frac{3}{3}\right) + 2f\left(\frac{4}{3}\right) + 4f\left(\frac{5}{3}\right) + f(2)\right] \approx 6.403292$

$E_T = I - T_6 \approx 6.4 - 6.695473 = -0.295473$

$E_M \approx 6.4 - 6.252572 = 0.147428$

$E_S \approx 6.4 - 6.403292 = -0.003292$

$n = 12$: $T_{12} = \frac{2}{12\cdot 2}\left\{f(0) + 2\left[f\left(\frac{1}{6}\right) + f\left(\frac{2}{6}\right) + f\left(\frac{3}{6}\right) + \cdots + f\left(\frac{11}{6}\right)\right] + f(2)\right\} \approx 6.474023$

$M_6 = \frac{2}{12}\left[f\left(\frac{1}{12}\right) + f\left(\frac{3}{12}\right) + f\left(\frac{5}{12}\right) + \cdots + f\left(\frac{23}{12}\right)\right] \approx 6.363008$

$S_6 = \frac{2}{12\cdot 3}\left[f(0) + 4f\left(\frac{1}{6}\right) + 2f\left(\frac{2}{6}\right) + 4f\left(\frac{3}{6}\right) + 2f\left(\frac{4}{6}\right) + \cdots + 4f\left(\frac{11}{6}\right) + f(2)\right] \approx 6.400206$

$E_T = I - T_{12} \approx 6.4 - 6.474023 = -0.074023$

$E_M \approx 6.4 - 6.363008 = 0.036992$

$E_S \approx 6.4 - 6.400206 = -0.000206$

| $n$ | $T_n$ | $M_n$ | $S_n$ |
|---|---|---|---|
| 6 | 6.695473 | 6.252572 | 6.403292 |
| 12 | 6.474023 | 6.363008 | 6.400206 |

| $n$ | $E_T$ | $E_M$ | $E_S$ |
|---|---|---|---|
| 6 | −0.295473 | 0.147428 | −0.003292 |
| 12 | −0.074023 | 0.036992 | −0.000206 |

*Observations:*

1. $E_T$ and $E_M$ are opposite in sign and decrease by a factor of about 4 as $n$ is doubled.
2. The Simpson's approximation is much more accurate than the Midpoint and Trapezoidal approximations, and $E_S$ seems to decrease by a factor of about 16 as $n$ is doubled.

**28.** $I = \int_1^4 \frac{1}{\sqrt{x}}\,dx = \left[2\sqrt{x}\right]_1^4 = 4 - 2 = 2$, $f(x) = \frac{1}{\sqrt{x}}$, $\Delta x = \frac{4-1}{n} = \frac{3}{n}$

$n = 6$: $T_6 = \frac{3}{6 \cdot 2}\left\{f(1) + 2\left[f\left(\frac{3}{2}\right) + f\left(\frac{4}{2}\right) + f\left(\frac{5}{2}\right) + f\left(\frac{6}{2}\right) + f\left(\frac{7}{2}\right)\right] + f(4)\right\} \approx 2.008966$

$M_6 = \frac{3}{6}\left[f\left(\frac{5}{4}\right) + f\left(\frac{7}{4}\right) + f\left(\frac{9}{4}\right) + f\left(\frac{11}{4}\right) + f\left(\frac{13}{4}\right) + f\left(\frac{15}{4}\right)\right] \approx 1.995572$

$S_6 = \frac{3}{6 \cdot 3}\left[f(1) + 4f\left(\frac{3}{2}\right) + 2f\left(\frac{4}{2}\right) + 4f\left(\frac{5}{2}\right) + 2f\left(\frac{6}{2}\right) + 4f\left(\frac{7}{2}\right) + f(4)\right] \approx 2.000469$

$E_T = I - T_6 \approx 2 - 2.008966 = -0.008966$,

$E_M \approx 2 - 1.995572 = 0.004428$,

$E_S \approx 2 - 2.000469 = -0.000469$

$n = 12$: $T_{12} = \frac{3}{12 \cdot 2}\left\{f(1) + 2\left[f\left(\frac{5}{4}\right) + f\left(\frac{6}{4}\right) + f\left(\frac{7}{4}\right) + \cdots + f\left(\frac{15}{4}\right)\right] + f(4)\right\} \approx 2.002269$

$M_{12} = \frac{3}{12}\left[f\left(\frac{9}{8}\right) + f\left(\frac{11}{8}\right) + f\left(\frac{13}{8}\right) + \cdots + f\left(\frac{31}{8}\right)\right] \approx 1.998869$

$S_{12} = \frac{3}{12 \cdot 3}\left[f(1) + 4f\left(\frac{5}{4}\right) + 2f\left(\frac{6}{4}\right) + 4f\left(\frac{7}{4}\right) + 2f\left(\frac{8}{4}\right) + \cdots + 4f\left(\frac{15}{4}\right) + f(4)\right] \approx 2.000036$

$E_T = I - T_{12} \approx 2 - 2.002269 = -0.002269$

$E_M \approx 2 - 1.998869 = 0.001131$

$E_S \approx 2 - 2.000036 = -0.000036$

| $n$ | $T_n$ | $M_n$ | $S_n$ |
|---|---|---|---|
| 6 | 2.008966 | 1.995572 | 2.000469 |
| 12 | 2.002269 | 1.998869 | 2.000036 |

| $n$ | $E_T$ | $E_M$ | $E_S$ |
|---|---|---|---|
| 6 | $-0.008966$ | 0.004428 | $-0.000469$ |
| 12 | $-0.002269$ | 0.001131 | $-0.000036$ |

*Observations:*

1. $E_T$ and $E_M$ are opposite in sign and decrease by a factor of about 4 as $n$ is doubled.
2. The Simpson's approximation is much more accurate than the Midpoint and Trapezoidal approximations, and $E_S$ seems decrease by a factor of about 16 as $n$ is doubled.

**29.** $\Delta x = (b-a)/n = (6-0)/6 = 1$

(a) $T_6 = \frac{\Delta x}{2}[f(0) + 2f(1) + 2f(2) + 2f(3) + 2f(4) + 2f(5) + f(6)]$

$\approx \frac{1}{2}[3 + 2(5) + 2(4) + 2(2) + 2(2.8) + 2(4) + 1]$

$= \frac{1}{2}(39.6) = 19.8$

(b) $M_6 = \Delta x[f(0.5) + f(1.5) + f(2.5) + f(3.5) + f(4.5) + f(5.5)]$

$\approx 1[4.5 + 4.7 + 2.6 + 2.2 + 3.4 + 3.2]$

$= 20.6$

(c) $S_6 = \frac{\Delta x}{3}[f(0) + 4f(1) + 2f(2) + 4f(3) + 2f(4) + 4f(5) + f(6)]$

$\approx \frac{1}{3}[3 + 4(5) + 2(4) + 4(2) + 2(2.8) + 4(4) + 1]$

$= \frac{1}{3}(61.6) = 20.5\overline{3}$

**30.** If $x$ = distance from left end of pool and $w = w(x)$ = width at $x$, then Simpson's Rule with $n = 8$ and $\Delta x = 2$ gives

Area $= \int_0^{16} w\,dx \approx \frac{2}{3}[0 + 4(6.2) + 2(7.2) + 4(6.8) + 2(5.6) + 4(5.0) + 2(4.8) + 4(4.8) + 0] \approx 84\text{ m}^2$.

**31.** (a) $\int_1^5 f(x)\,dx \approx M_4 = \frac{5-1}{4}[f(1.5) + f(2.5) + f(3.5) + f(4.5)] = 1(2.9 + 3.6 + 4.0 + 3.9) = 14.4$

(b) $-2 \le f''(x) \le 3 \;\Rightarrow\; |f''(x)| \le 3 \;\Rightarrow\; K = 3$, since $|f''(x)| \le K$. The error estimate for the Midpoint Rule is

$$|E_M| \le \frac{K(b-a)^3}{24n^2} = \frac{3(5-1)^3}{24(4)^2} = \frac{1}{2}.$$

**32.**

(a) $\int_0^{1.6} g(x)\,dx \approx S_8 = \frac{1.6-0}{8\cdot 3}[g(0) + 4g(0.2) + 2g(0.4) + 4g(0.6) + 2g(0.8) + 4g(1.0) + 2g(1.2) + 4g(1.4) + g(1.6)]$

$= \frac{1}{15}[12.1 + 4(11.6) + 2(11.3) + 4(11.1) + 2(11.7) + 4(12.2) + 2(12.6) + 4(13.0) + 13.2]$

$= \frac{1}{15}(288.1) = \frac{2881}{150} \approx 19.2$

(b) $-5 \le g^{(4)}(x) \le 2 \;\Rightarrow\; \left|g^{(4)}(x)\right| \le 5 \;\Rightarrow\; K = 5$, since $\left|g^{(4)}(x)\right| \le K$. The error estimate for Simpson's Rule is

$$|E_S| \le \frac{K(b-a)^5}{180n^4} = \frac{5(1.6-0)^5}{180(8)^4} = \frac{2}{28{,}125} = 7.\overline{1} \times 10^{-5}.$$

**33.** $T_{\text{ave}} = \frac{1}{24-0}\int_0^{24} T(t)\,dt \approx \frac{1}{24}S_{12} = \frac{1}{24}\,\frac{24-0}{3(12)}[T(0) + 4T(2) + 2T(4) + 4T(6) + 2T(8) + 4T(10) + 2T(12)$

$+ 4T(14) + 2T(16) + 4T(18) + 2T(20) + 4T(22) + T(24)]$

$\approx \frac{1}{36}[67 + 4(65) + 2(62) + 4(58) + 2(56) + 4(61) + 2(63) + 4(68)$

$+ 2(71) + 4(69) + 2(67) + 4(66) + 64]$

$= \frac{1}{36}(2317) = 64.36\overline{1}^\circ\text{F}.$

The average temperature was about $64.4^\circ$F.

**34.** We use Simpson's Rule with $n = 10$ and $\Delta x = \frac{1}{2}$:

$$\begin{aligned} \text{distance} = \int_0^5 v(t)\,dt \approx S_{10} &= \tfrac{1}{2\cdot 3}[f(0) + 4f(0.5) + 2f(1) + \cdots + 4f(4.5) + f(5)] \\ &= \tfrac{1}{6}[0 + 4(4.67) + 2(7.34) + 4(8.86) + 2(9.73) + 4(10.22) \\ &\qquad + 2(10.51) + 4(10.67) + 2(10.76) + 4(10.81) + 10.81] \\ &= \tfrac{1}{6}(268.41) = 44.735 \text{ m} \end{aligned}$$

**35.** By the Net Change Theorem, the increase in velocity is equal to $\int_0^6 a(t)\,dt$. We use Simpson's Rule with $n = 6$ and $\Delta t = (6 - 0)/6 = 1$ to estimate this integral:

$$\begin{aligned} \int_0^6 a(t)\,dt \approx S_6 &= \tfrac{1}{3}[a(0) + 4a(1) + 2a(2) + 4a(3) + 2a(4) + 4a(5) + a(6)] \\ &\approx \tfrac{1}{3}[0 + 4(0.5) + 2(4.1) + 4(9.8) + 2(12.9) + 4(9.5) + 0] = \tfrac{1}{3}(113.2) = 37.7\overline{3} \text{ ft/s} \end{aligned}$$

**36.** By the Net Change Theorem, the total amount of water that leaked out during the first six hours is equal to $\int_0^6 r(t)\,dt$.

We use Simpson's Rule with $n = 6$ and $\Delta t = \frac{6-0}{6} = 1$ to estimate this integral:

$$\begin{aligned} \int_0^6 r(t)\,dt \approx S_6 &= \tfrac{1}{3}[r(0) + 4r(1) + 2r(2) + 4r(3) + 2r(4) + 4r(5) + r(6)] \\ &\approx \tfrac{1}{3}[4 + 4(3) + 2(2.4) + 4(1.9) + 2(1.4) + 4(1.1) + 1] = \tfrac{1}{3}(36.6) = 12.2 \text{ liters} \end{aligned}$$

**37.** By the Net Change Theorem, the energy used is equal to $\int_0^6 P(t)\,dt$. We use Simpson's Rule with $n = 12$ and $\Delta t = \frac{6-0}{12} = \frac{1}{2}$ to estimate this integral:

$$\begin{aligned} \int_0^6 P(t)\,dt \approx S_{12} &= \tfrac{1/2}{3}[P(0) + 4P(0.5) + 2P(1) + 4P(1.5) + 2P(2) + 4P(2.5) + 2P(3) \\ &\qquad + 4P(3.5) + 2P(4) + 4P(4.5) + 2P(5) + 4P(5.5) + P(6)] \\ &= \tfrac{1}{6}[1814 + 4(1735) + 2(1686) + 4(1646) + 2(1637) + 4(1609) + 2(1604) \\ &\qquad + 4(1611) + 2(1621) + 4(1666) + 2(1745) + 4(1886) + 2052] \\ &= \tfrac{1}{6}(61{,}064) = 10{,}177.\overline{3} \text{ megawatt-hours} \end{aligned}$$

**38.**

By the Net Change Theorem, the total amount of data transmitted is equal to $\int_0^8 D(t)\,dt \times 3600$ [since $D(t)$ is measured in megabits per second and $t$ is in hours]. We use Simpson's Rule with $n = 8$ and $\Delta t = (8 - 0)/8 = 1$ to estimate this integral:

$$\begin{aligned} \int_0^8 D(t)\,dt \approx S_8 &= \tfrac{1}{3}[D(0) + 4D(1) + 2D(2) + 4D(3) + 2D(4) + 4D(5) + 2D(6) + 4D(7) + D(8)] \\ &\approx \tfrac{1}{3}[0.35 + 4(0.32) + 2(0.41) + 4(0.50) + 2(0.51) + 4(0.56) + 2(0.56) + 4(0.83) + 0.88] \\ &= \tfrac{1}{3}(13.03) = 4.34\overline{3} \end{aligned}$$

Now multiply by 3600 to obtain 15,636 megabits.

**39.** (a) Let $y = f(x)$ denote the curve. Using disks, $V = \int_2^{10} \pi[f(x)]^2\,dx = \pi\int_2^{10} g(x)\,dx = \pi I_1$.

Now use Simpson's Rule to approximate $I_1$:

$$\begin{aligned} I_1 \approx S_8 &= \tfrac{10-2}{3(8)}[g(2) + 4g(3) + 2g(4) + 4g(5) + 2g(6) + 4g(7) + g(8)] \\ &\approx \tfrac{1}{3}[0^2 + 4(1.5)^2 + 2(1.9)^2 + 4(2.2)^2 + 2(3.0)^2 + 4(3.8)^2 + 2(4.0)^2 + 4(3.1)^2 + 0^2] \\ &= \tfrac{1}{3}(181.78) \end{aligned}$$

Thus, $V \approx \pi \cdot \frac{1}{3}(181.78) \approx 190.4$ or 190 cubic units.

(b) Using cylindrical shells, $V = \int_2^{10} 2\pi x f(x)\,dx = 2\pi\int_2^{10} x f(x)\,dx = 2\pi I_1$.

Now use Simpson's Rule to approximate $I_1$:

$$\begin{aligned} I_1 \approx S_8 &= \tfrac{10-2}{3(8)}[2f(2) + 4\cdot 3f(3) + 2\cdot 4f(4) + 4\cdot 5f(5) + 2\cdot 6f(6) \\ &\qquad + 4\cdot 7f(7) + 2\cdot 8f(8) + 4\cdot 9f(9) + 10f(10)] \\ &\approx \tfrac{1}{3}[2(0) + 12(1.5) + 8(1.9) + 20(2.2) + 12(3.0) + 28(3.8) + 16(4.0) + 36(3.1) + 10(0)] \\ &= \tfrac{1}{3}(395.2) \end{aligned}$$

Thus, $V \approx 2\pi \cdot \frac{1}{3}(395.2) \approx 827.7$ or 828 cubic units.

**40.** Work $= \int_0^{18} f(x)\,dx \approx S_6 = \frac{18-0}{6\cdot 3}[f(0) + 4f(3) + 2f(6) + 4f(9) + 2f(12) + 4f(15) + f(18)]$

$$= 1\cdot[9.8 + 4(9.1) + 2(8.5) + 4(8.0) + 2(7.7) + 4(7.5) + 7.4] = 148 \text{ joules}$$

**41.** Using disks, $V = \int_1^5 \pi(e^{-1/x})^2\,dx = \pi\int_1^5 e^{-2/x}\,dx = \pi I_1$. Now use Simpson's Rule with $f(x) = e^{-2/x}$ to approximate $I_1$. $I_1 \approx S_8 = \frac{5-1}{3(8)}[f(1) + 4f(1.5) + 2f(2) + 4f(2.5) + 2f(3) + 4f(3.5) + 2f(4) + 4f(4.5) + f(5)] \approx \frac{1}{6}(11.4566)$

Thus, $V \approx \pi \cdot \frac{1}{6}(11.4566) \approx 6.0$ cubic units.

**42.** Using Simpson's Rule with $n = 10$, $\Delta x = \frac{\pi/2}{10}$, $L = 1$, $\theta_0 = \frac{42\pi}{180}$ radians, $g = 9.8\text{ m/s}^2$, $k^2 = \sin^2\left(\frac{1}{2}\theta_0\right)$, and $f(x) = 1/\sqrt{1 - k^2\sin^2 x}$, we get

$$\begin{aligned} T &= 4\sqrt{\frac{L}{g}}\int_0^{\pi/2}\frac{dx}{\sqrt{1 - k^2\sin^2 x}} \approx 4\sqrt{\frac{L}{g}}\,S_{10} \\ &= 4\sqrt{\tfrac{1}{9.8}}\left(\tfrac{\pi/2}{10\cdot 3}\right)\left[f(0) + 4f\left(\tfrac{\pi}{20}\right) + 2f\left(\tfrac{2\pi}{20}\right) + \cdots + 4f\left(\tfrac{9\pi}{20}\right) + f\left(\tfrac{\pi}{2}\right)\right] \approx 2.07665 \end{aligned}$$

**43.** $I(\theta) = \dfrac{N^2\sin^2 k}{k^2}$, where $k = \dfrac{\pi N d\sin\theta}{\lambda}$, $N = 10{,}000$, $d = 10^{-4}$, and $\lambda = 632.8\times 10^{-9}$. So $I(\theta) = \dfrac{(10^4)^2\sin^2 k}{k^2}$, where $k = \dfrac{\pi(10^4)(10^{-4})\sin\theta}{632.8\times 10^{-9}}$. Now $n = 10$ and $\Delta\theta = \dfrac{10^{-6} - (-10^{-6})}{10} = 2\times 10^{-7}$, so

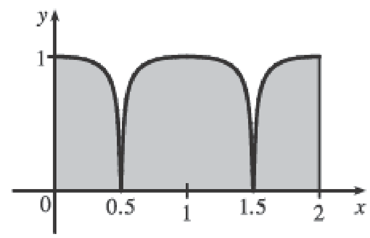
$M_{10} = 2\times 10^{-7}[I(-0.0000009) + I(-0.0000007) + \cdots + I(0.0000009)] \approx 59.4$.

**44.** $f(x) = \cos(\pi x)$, $\Delta x = \frac{20-0}{10} = 2 \quad \Rightarrow$

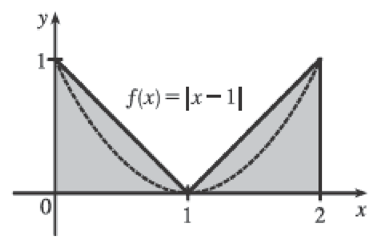
$$T_{10} = \tfrac{2}{2}\{f(0) + 2[f(2) + f(4) + \cdots + f(18)] + f(20)\} = 1[\cos 0 + 2(\cos 2\pi + \cos 4\pi + \cdots + \cos 18\pi) + \cos 20\pi]$$
$$= 1 + 2(1 + 1 + 1 + 1 + 1 + 1 + 1 + 1 + 1) + 1 = 20$$

The actual value is $\int_0^{20} \cos(\pi x)\,dx = \frac{1}{\pi}\left[\sin \pi x\right]_0^{20} = \frac{1}{\pi}(\sin 20\pi - \sin 0) = 0$. The discrepancy is due to the fact that the function is sampled only at points of the form $2n$, where its value is $f(2n) = \cos(2n\pi) = 1$.

**45.** Consider the function $f$ whose graph is shown. The area $\int_0^2 f(x)\,dx$ is close to 2. The Trapezoidal Rule gives
$T_2 = \frac{2-0}{2 \cdot 2}\,[f(0) + 2f(1) + f(2)] = \frac{1}{2}\,[1 + 2 \cdot 1 + 1] = 2.$
The Midpoint Rule gives $M_2 = \frac{2-0}{2}\,[f(0.5) + f(1.5)] = 1[0 + 0] = 0$, so the Trapezoidal Rule is more accurate.




**46.** Consider the function $f(x) = |x - 1|$, $0 \le x \le 2$. The area $\int_0^2 f(x)\,dx$ is exactly 1. So is the right endpoint approximation:
$R_2 = f(1)\,\Delta x + f(2)\,\Delta x = 0 \cdot 1 + 1 \cdot 1 = 1$. But Simpson's Rule approximates $f$ with the parabola $y = (x-1)^2$, shown dashed, and
$$S_2 = \frac{\Delta x}{3}\,[f(0) + 4f(1) + f(2)] = \frac{1}{3}\,[1 + 4 \cdot 0 + 1] = \frac{2}{3}.$$




**47.**
Since the Trapezoidal and Midpoint approximations on the interval $[a, b]$ are the sums of the Trapezoidal and Midpoint approximations on the subintervals $[x_{i-1}, x_i]$, $i = 1, 2, \ldots, n$, we can focus our attention on one such interval. The condition $f''(x) < 0$ for $a \le x \le b$ means that the graph of $f$ is concave down as in Figure 5. In that figure, $T_n$ is the area of the trapezoid $AQRD$, $\int_a^b f(x)\,dx$ is the area of the region $AQPRD$, and $M_n$ is the area of the trapezoid $ABCD$, so $T_n < \int_a^b f(x)\,dx < M_n$. In general, the condition $f'' < 0$ implies that the graph of $f$ on $[a, b]$ lies above the chord joining the points $(a, f(a))$ and $(b, f(b))$. Thus, $\int_a^b f(x)\,dx > T_n$. Since $M_n$ is the area under a tangent to the graph, and since $f'' < 0$ implies that the tangent lies above the graph, we also have $M_n > \int_a^b f(x)\,dx$. Thus, $T_n < \int_a^b f(x)\,dx < M_n$.

**48.**

Let $f$ be a polynomial of degree $\leq 3$; say $f(x) = Ax^3 + Bx^2 + Cx + D$. It will suffice to show that Simpson's estimate is exact when there are two subintervals ($n = 2$), because for a larger even number of subintervals the sum of exact estimates is exact. As in the derivation of Simpson's Rule, we can assume that $x_0 = -h$, $x_1 = 0$, and $x_2 = h$. Then Simpson's approximation is

$$\int_{-h}^{h} f(x)\,dx \approx \tfrac{1}{3}h[f(-h) + 4f(0) + f(h)] = \tfrac{1}{3}h\left[\left(-Ah^3 + Bh^2 - Ch + D\right) + 4D + \left(Ah^3 + Bh^2 + Ch + D\right)\right]$$
$$= \tfrac{1}{3}h[2Bh^2 + 6D] = \tfrac{2}{3}Bh^3 + 2Dh$$

The exact value of the integral is

$$\int_{-h}^{h} (Ax^3 + Bx^2 + Cx + D)\,dx = 2\int_0^h (Bx^2 + D)\,dx \qquad \text{[by Theorem 4.5.6(a) and (b)]}$$
$$= 2\left[\tfrac{1}{3}Bx^3 + Dx\right]_0^h = \tfrac{2}{3}Bh^3 + 2Dh$$

Thus, Simpson's Rule is exact.