1.
Let $u = \sin x$, so that $du = \cos x\,dx$. Then $\int \cos x(1+\sin^2 x)\,dx = \int (1+u^2)\,du = u + \frac{1}{3}u^3 + C = \sin x + \frac{1}{3}\sin^3 x + C$.

2. Let $u = 3x+1$. Then $du = 3\,dx \quad \Rightarrow$

$$\int_0^1 (3x+1)^{\sqrt{2}}\,dx = \int_1^4 u^{\sqrt{2}}\left(\frac{1}{3}\,du\right) = \frac{1}{3}\left[\frac{1}{\sqrt{2}+1}\,u^{\sqrt{2}+1}\right]_1^4 = \frac{1}{3(\sqrt{2}+1)}\left(4^{\sqrt{2}+1}-1\right)$$

3. $$\int \frac{\sin x + \sec x}{\tan x}\,dx = \int\left(\frac{\sin x}{\tan x} + \frac{\sec x}{\tan x}\right)dx = \int (\cos x + \csc x)\,dx = \sin x + \ln|\csc x - \cot x| + C$$

4. $$\int \frac{\sin^3 x}{\cos x}\,dx = \int \frac{\sin^2 x\,\sin x}{\cos x}\,dx = \int \frac{(1-\cos^2 x)\sin x}{\cos x}\,dx = \int \frac{1-u^2}{u}\,(-du) \qquad \begin{bmatrix} u = \cos x \\ du = -\sin x\,dx \end{bmatrix}$$

$$= \int \left(u - \tfrac{1}{u}\right)du = \tfrac{1}{2}u^2 - \ln|u| + C = \tfrac{1}{2}\cos^2 x - \ln|\cos x| + C$$

5. Let $u = t^2$. Then $du = 2t\,dt \quad \Rightarrow$

$$\int \frac{t}{t^4+2}\,dt = \int \frac{1}{u^2+2}\left(\frac{1}{2}\,du\right) = \frac{1}{2}\frac{1}{\sqrt{2}}\tan^{-1}\left(\frac{u}{\sqrt{2}}\right) + C \;\text{[by Formula 17]}\; = \frac{1}{2\sqrt{2}}\tan^{-1}\left(\frac{t^2}{\sqrt{2}}\right) + C$$

6. Let $u = 2x+1$. Then $du = 2\,dx \quad \Rightarrow$

$$\int_0^1 \frac{x}{(2x+1)^3}\,dx = \int_1^3 \frac{(u-1)/2}{u^3}\left(\frac{1}{2}\,du\right) = \frac{1}{4}\int_1^3\left(\frac{1}{u^2} - \frac{1}{u^3}\right)du = \frac{1}{4}\left[-\frac{1}{u} + \frac{1}{2u^2}\right]_1^3$$

$$= \tfrac{1}{4}\left[\left(-\tfrac{1}{3} + \tfrac{1}{18}\right) - \left(-1 + \tfrac{1}{2}\right)\right] = \tfrac{1}{4}\left(\tfrac{2}{9}\right) = \tfrac{1}{18}$$

7. Let $u = \arctan y$. Then $du = \dfrac{dy}{1+y^2} \quad \Rightarrow \quad \displaystyle\int_{-1}^{1} \frac{e^{\arctan y}}{1+y^2}\,dy = \int_{-\pi/4}^{\pi/4} e^u\,du = \left[e^u\right]_{-\pi/4}^{\pi/4} = e^{\pi/4} - e^{-\pi/4}.$

8. $$\int t\sin t\,\cos t\,dt = \int t\cdot\tfrac{1}{2}(2\sin t\,\cos t)\,dt = \tfrac{1}{2}\int t\sin 2t\,dt$$

$$= \tfrac{1}{2}\left(-\tfrac{1}{2}t\cos 2t - \int -\tfrac{1}{2}\cos 2t\,dt\right) \qquad \begin{bmatrix} u = t, & dv = \sin 2t\,dt \\ du = dt, & v = -\tfrac{1}{2}\cos 2t \end{bmatrix}$$

$$= -\tfrac{1}{4}t\cos 2t + \tfrac{1}{4}\int \cos 2t\,dt = -\tfrac{1}{4}t\cos 2t + \tfrac{1}{8}\sin 2t + C$$

9. $$\int_1^3 r^4 \ln r\,dr \quad \begin{bmatrix} u = \ln r, & dv = r^4\,dr, \\ du = \frac{dr}{r} & v = \frac{1}{5}r^5 \end{bmatrix} = \left[\tfrac{1}{5}r^5\ln r\right]_1^3 - \int_1^3 \tfrac{1}{5}r^4\,dr = \tfrac{243}{5}\ln 3 - 0 - \left[\tfrac{1}{25}r^5\right]_1^3$$

$$= \tfrac{243}{5}\ln 3 - \left(\tfrac{243}{25} - \tfrac{1}{25}\right) = \tfrac{243}{5}\ln 3 - \tfrac{242}{25}$$

10. $\dfrac{x-1}{x^2-4x-5} = \dfrac{x-1}{(x-5)(x+1)} = \dfrac{A}{x-5} + \dfrac{B}{x+1} \quad \Rightarrow \quad x-1 = A(x+1) + B(x-5)$. Setting $x=-1$ gives $-2=-6B$, so $B=\frac{1}{3}$. Setting $x=5$ gives $4=6A$, so $A=\frac{2}{3}$. Now

$$\int_0^4 \frac{x-1}{x^2-4x-5}\,dx = \int_0^4 \left(\frac{2/3}{x-5} + \frac{1/3}{x+1}\right) dx = \left[\tfrac{2}{3}\ln|x-5| + \tfrac{1}{3}\ln|x+1|\right]_0^4$$
$$= \tfrac{2}{3}\ln 1 + \tfrac{1}{3}\ln 5 - \tfrac{2}{3}\ln 5 - \tfrac{1}{3}\ln 1 = -\tfrac{1}{3}\ln 5$$

11. $\displaystyle\int \frac{x-1}{x^2-4x+5}\,dx = \int \frac{(x-2)+1}{(x-2)^2+1}\,dx = \int \left(\frac{u}{u^2+1} + \frac{1}{u^2+1}\right) du \qquad [u=x-2,\, du=dx]$

$$= \tfrac{1}{2}\ln(u^2+1) + \tan^{-1}u + C = \tfrac{1}{2}\ln(x^2-4x+5) + \tan^{-1}(x-2) + C$$

12. $\displaystyle\int \frac{x}{x^4+x^2+1}\,dx = \int \frac{\frac{1}{2}\,du}{u^2+u+1} \quad \begin{bmatrix} u=x^2, \\ du = 2x\,dx \end{bmatrix} \quad = \frac{1}{2}\int \frac{du}{\left(u+\frac{1}{2}\right)^2 + \frac{3}{4}}$

$$= \frac{1}{2}\int \frac{\frac{\sqrt{3}}{2}\,dv}{\frac{3}{4}(v^2+1)} \quad \begin{bmatrix} u+\frac{1}{2} = \frac{\sqrt{3}}{2}v, \\ du = \frac{\sqrt{3}}{2}\,dv \end{bmatrix} \quad = \frac{\sqrt{3}}{4}\cdot\frac{4}{3}\int \frac{dv}{v^2+1}$$
$$= \tfrac{1}{\sqrt{3}}\tan^{-1}v + C = \tfrac{1}{\sqrt{3}}\tan^{-1}\left(\tfrac{2}{\sqrt{3}}\left(x^2+\tfrac{1}{2}\right)\right) + C$$

13. $\int \sin^5 t\,\cos^4 t\,dt = \int \sin^4 t\,\cos^4 t\,\sin t\,dt = \int (\sin^2 t)^2\cos^4 t\,\sin t\,dt$

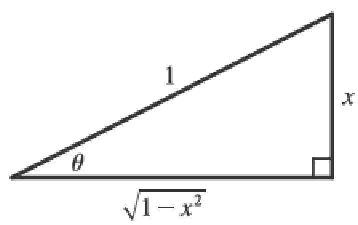
$$= \textstyle\int (1-\cos^2 t)^2\cos^4 t\,\sin t\,dt = \int (1-u^2)^2 u^4\,(-du) \quad [u=\cos t,\, du=-\sin t\,dt]$$
$$= \textstyle\int (-u^4+2u^6-u^8)\,du = -\frac{1}{5}u^5 + \frac{2}{7}u^7 - \frac{1}{9}u^9 + C = -\frac{1}{5}\cos^5 t + \frac{2}{7}\cos^7 t - \frac{1}{9}\cos^9 t + C$$

14. Let $u=1+x^2$, so that $du=2x\,dx$. Then

$$\int \frac{x^3}{\sqrt{1+x^2}}\,dx = \int \frac{x^2}{\sqrt{1+x^2}}\,(x\,dx) = \int \frac{u-1}{u^{1/2}}\left(\tfrac{1}{2}\,du\right) = \frac{1}{2}\int (u^{1/2}-u^{-1/2})\,du = \frac{1}{2}\left(\frac{2}{3}u^{3/2} - 2u^{1/2}\right) + C$$
$$= \tfrac{1}{3}(1+x^2)^{3/2} - (1+x^2)^{1/2} + C \quad \left[\text{or } \tfrac{1}{3}(x^2-2)\sqrt{1+x^2} + C\right]$$

15. Let $x=\sin\theta$, where $-\frac{\pi}{2} \le \theta \le \frac{\pi}{2}$. Then $dx=\cos\theta\,d\theta$ and $(1-x^2)^{1/2} = \cos\theta$, so

$$\int \frac{dx}{(1-x^2)^{3/2}} = \int \frac{\cos\theta\,d\theta}{(\cos\theta)^3} = \int \sec^2\theta\,d\theta = \tan\theta + C = \frac{x}{\sqrt{1-x^2}} + C.$$



16. $\displaystyle\int_0^{\sqrt{2}/2} \frac{x^2}{\sqrt{1-x^2}}\,dx = \int_0^{\pi/4} \frac{\sin^2\theta}{\cos\theta}\cos\theta\,d\theta \qquad \begin{bmatrix} u=\sin\theta, \\ du=\cos\theta\,d\theta \end{bmatrix}$

$$= \textstyle\int_0^{\pi/4} \frac{1}{2}(1-\cos 2\theta)\,d\theta = \frac{1}{2}\left[\theta - \frac{1}{2}\sin 2\theta\right]_0^{\pi/4} = \frac{1}{2}\left[\left(\frac{\pi}{4}-\frac{1}{2}\right) - (0-0)\right] = \frac{\pi}{8} - \frac{1}{4}$$

17. $\int_0^\pi t\,\cos^2 t\,dt = \int_0^\pi t\left[\frac{1}{2}(1+\cos 2t)\right]dt = \frac{1}{2}\int_0^\pi t\,dt + \frac{1}{2}\int_0^\pi t\cos 2t\,dt$

$= \frac{1}{2}\left[\frac{1}{2}t^2\right]_0^\pi + \frac{1}{2}\left[\frac{1}{2}t\sin 2t\right]_0^\pi - \frac{1}{2}\int_0^\pi \frac{1}{2}\sin 2t\,dt \qquad \begin{bmatrix} u=t, & dv=\cos 2t\,dt \\ du=dt, & v=\frac{1}{2}\sin 2t \end{bmatrix}$

$= \frac{1}{4}\pi^2 + 0 - \frac{1}{4}\left[-\frac{1}{2}\cos 2t\right]_0^\pi = \frac{1}{4}\pi^2 + \frac{1}{8}(1-1) = \frac{1}{4}\pi^2$

18. Let $u=\sqrt{t}$. Then $du = \dfrac{1}{2\sqrt{t}}\,dt \quad\Rightarrow\quad \displaystyle\int_1^4 \frac{e^{\sqrt{t}}}{\sqrt{t}}\,dt = \int_1^2 e^u\,(2\,du) = 2\left[e^u\right]_1^2 = 2(e^2-e)$.

19. Let $u=e^x$. Then $\int e^{x+e^x}\,dx = \int e^{e^x}e^x\,dx = \int e^u\,du = e^u + C = e^{e^x}+C$.

20. Since e^2 is a constant, $\int e^2\,dx = e^2x + C$.

21. Let $t=\sqrt{x}$, so that $t^2=x$ and $2t\,dt = dx$. Then $\int \arctan\sqrt{x}\,dx = \int \arctan t\,(2t\,dt) = I$. Now use parts with

$u=\arctan t,\ dv = 2t\,dt \quad\Rightarrow\quad du = \dfrac{1}{1+t^2}\,dt,\ v=t^2$. Thus,

$$I = t^2\arctan t - \int \frac{t^2}{1+t^2}\,dt = t^2\arctan t - \int\left(1-\frac{1}{1+t^2}\right)dt = t^2\arctan t - t + \arctan t + C$$

$$= x\arctan\sqrt{x} - \sqrt{x} + \arctan\sqrt{x} + C \quad \left[\text{or } (x+1)\arctan\sqrt{x} - \sqrt{x} + C\right]$$

22. Let $u = 1+(\ln x)^2$, so that $du = \dfrac{2\ln x}{x}\,dx$. Then

$$\int \frac{\ln x}{x\sqrt{1+(\ln x)^2}}\,dx = \frac{1}{2}\int\frac{1}{\sqrt{u}}\,du = \frac{1}{2}\left(2\sqrt{u}\right) + C = \sqrt{1+(\ln x)^2}+C.$$

23. Let $u = 1+\sqrt{x}$. Then $x=(u-1)^2$, $dx = 2(u-1)\,du \quad\Rightarrow$

$$\int_0^1\left(1+\sqrt{x}\right)^8 dx = \int_1^2 u^8\cdot 2(u-1)\,du = 2\int_1^2(u^9-u^8)\,du = \left[\tfrac{1}{5}u^{10} - 2\cdot\tfrac{1}{9}u^9\right]_1^2 = \tfrac{1024}{5} - \tfrac{1024}{9} - \tfrac{1}{5} + \tfrac{2}{9} = \tfrac{4097}{45}.$$

24. $\displaystyle\int_0^4 \frac{6z+5}{2z+1}\,dz = \int_0^4\frac{6z+3+2}{2z+1}\,dz = \int_0^4\frac{3(2z+1)+2}{2z+1}\,dz = \int_0^4\left(3+\frac{2}{2z+1}\right)dz$

$= \Big[3z + \ln|2z+1|\Big]_0^4 = 12 + \ln 9$

25. $\dfrac{3x^2-2}{x^2-2x-8} = 3 + \dfrac{6x+22}{(x-4)(x+2)} = 3 + \dfrac{A}{x-4} + \dfrac{B}{x+2} \quad\Rightarrow\quad 6x+22 = A(x+2)+B(x-4)$. Setting

$x=4$ gives $46 = 6A$, so $A=\frac{23}{3}$. Setting $x=-2$ gives $10=-6B$, so $B=-\frac{5}{3}$. Now

$$\int\frac{3x^2-2}{x^2-2x-8}\,dx = \int\left(3+\frac{23/3}{x-4} - \frac{5/3}{x+2}\right)dx = 3x + \tfrac{23}{3}\ln|x-4| - \tfrac{5}{3}\ln|x+2| + C.$$

26. $\displaystyle\int \frac{3x^2-2}{x^3-2x-8}\,dx = \int \frac{du}{u} \quad \begin{bmatrix} u = x^3-2x-8, \\ du = (3x^2-2)\,dx \end{bmatrix} \quad = \ln|u| + C = \ln\left|x^3-2x-8\right| + C$

27. Let $u = 1 + e^x$, so that $du = e^x\,dx = (u-1)\,dx$. Then $\displaystyle\int \frac{1}{1+e^x}\,dx = \int \frac{1}{u}\cdot\frac{du}{u-1} = \int \frac{1}{u(u-1)}\,du = I$. Now

$\displaystyle\frac{1}{u(u-1)} = \frac{A}{u} + \frac{B}{u-1} \quad\Rightarrow\quad 1 = A(u-1) + Bu$. Set $u = 1$ to get $1 = B$. Set $u = 0$ to get $1 = -A$, so $A = -1$.

Thus, $\displaystyle I = \int\left(\frac{-1}{u} + \frac{1}{u-1}\right)du = -\ln|u| + \ln|u-1| + C = -\ln(1+e^x) + \ln e^x + C = x - \ln(1+e^x) + C.$

Another method: Multiply numerator and denominator by e^{-x} and let $u = e^{-x} + 1$. This gives the answer in the form $-\ln(e^{-x}+1) + C$.

28. $\int \sin\sqrt{at}\,dt = \int \sin u \cdot \frac{2}{a}u\,du \quad [u = \sqrt{at},\, u^2 = at,\, 2u\,du = a\,dt] \quad = \frac{2}{a}\int u\sin u\,du$

$$= \tfrac{2}{a}[-u\cos u + \sin u] + C \quad \text{[integration by parts]} \quad = -\tfrac{2}{a}\sqrt{at}\,\cos\sqrt{at} + \tfrac{2}{a}\,\sin\sqrt{at} + C$$

$$= -2\sqrt{\tfrac{t}{a}}\,\cos\sqrt{at} + \tfrac{2}{a}\,\sin\sqrt{at} + C$$

29. Use integration by parts with $u = \ln\left(x + \sqrt{x^2-1}\right)$, $dv = dx \quad\Rightarrow$

$\displaystyle du = \frac{1}{x+\sqrt{x^2-1}}\left(1 + \frac{x}{\sqrt{x^2-1}}\right)dx = \frac{1}{x+\sqrt{x^2-1}}\left(\frac{\sqrt{x^2-1}+x}{\sqrt{x^2-1}}\right)dx = \frac{1}{\sqrt{x^2-1}}\,dx$, $v = x$. Then

$$\int \ln\left(x+\sqrt{x^2-1}\right)dx = x\ln\left(x+\sqrt{x^2-1}\right) - \int \frac{x}{\sqrt{x^2-1}}\,dx = x\ln\left(x+\sqrt{x^2-1}\right) - \sqrt{x^2-1} + C.$$

30. $|e^x - 1| = \begin{cases} e^x - 1 & \text{if } e^x - 1 \ge 0 \\ -(e^x-1) & \text{if } e^x - 1 < 0 \end{cases} = \begin{cases} e^x - 1 & \text{if } x \ge 0 \\ 1 - e^x & \text{if } x < 0 \end{cases}$

Thus, $\int_{-1}^{2} |e^x - 1|\,dx = \int_{-1}^{0}(1-e^x)\,dx + \int_{0}^{2}(e^x-1)\,dx = \Big[x - e^x\Big]_{-1}^{0} + \Big[e^x - x\Big]_{0}^{2}$

$$= (0-1) - (-1-e^{-1}) + (e^2-2) - (1-0) = e^2 + e^{-1} - 3$$

31.
As in Example 5,

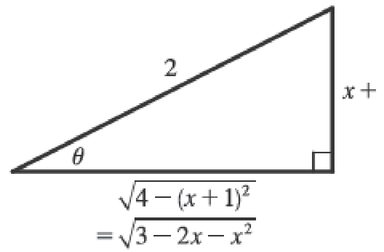
$$\int\sqrt{\frac{1+x}{1-x}}\,dx = \int\frac{\sqrt{1+x}}{\sqrt{1-x}}\cdot\frac{\sqrt{1+x}}{\sqrt{1+x}}\,dx = \int\frac{1+x}{\sqrt{1-x^2}}\,dx = \int\frac{dx}{\sqrt{1-x^2}} + \int\frac{x\,dx}{\sqrt{1-x^2}} = \sin^{-1}x - \sqrt{1-x^2} + C.$$

Another method: Substitute $u = \sqrt{(1+x)/(1-x)}$.

32. $\int \frac{\sqrt{2x-1}}{2x+3}\,dx = \int \frac{u \cdot u\,du}{u^2+4} \quad \left[\begin{matrix} u=\sqrt{2x-1},\, 2x+3=u^2+4, \\ u^2=2x-1,\, u\,du=dx \end{matrix}\right] \quad = \int \left(1 - \frac{4}{u^2+4}\right) du$

$= u - 4 \cdot \frac{1}{2}\tan^{-1}\left(\frac{1}{2}u\right) + C = \sqrt{2x-1} - 2\tan^{-1}\left(\frac{1}{2}\sqrt{2x-1}\right) + C$

33. $3-2x-x^2 = -(x^2+2x+1)+4 = 4-(x+1)^2$. Let $x+1=2\sin\theta$, where $-\frac{\pi}{2} \le \theta \le \frac{\pi}{2}$. Then $dx = 2\cos\theta\,d\theta$ and



$\int \sqrt{3-2x-x^2}\,dx = \int \sqrt{4-(x+1)^2}\,dx = \int \sqrt{4-4\sin^2\theta}\,2\cos\theta\,d\theta$

$= 4\int \cos^2\theta\,d\theta = 2\int(1+\cos 2\theta)\,d\theta$

$= 2\theta + \sin 2\theta + C = 2\theta + 2\sin\theta\cos\theta + C$

$= 2\sin^{-1}\left(\frac{x+1}{2}\right) + 2 \cdot \frac{x+1}{2} \cdot \frac{\sqrt{3-2x-x^2}}{2} + C$

$= 2\sin^{-1}\left(\frac{x+1}{2}\right) + \frac{x+1}{2}\sqrt{3-2x-x^2} + C$

34. $\int_{\pi/4}^{\pi/2} \frac{1+4\cot x}{4-\cot x}\,dx = \int_{\pi/4}^{\pi/2} \left[\frac{(1+4\cos x/\sin x)}{(4-\cos x/\sin x)} \cdot \frac{\sin x}{\sin x}\right] dx = \int_{\pi/4}^{\pi/2} \frac{\sin x + 4\cos x}{4\sin x - \cos x}\,dx$

$= \int_{3/\sqrt{2}}^{4} \frac{1}{u}\,du \quad \left[\begin{matrix} u = 4\sin x - \cos x, \\ du = (4\cos x + \sin x)\,dx \end{matrix}\right]$

$= \Big[\ln|u|\Big]_{3/\sqrt{2}}^{4} = \ln 4 - \ln\frac{3}{\sqrt{2}} = \ln\frac{4}{3/\sqrt{2}} = \ln\left(\frac{4}{3}\sqrt{2}\right)$

35. Using product formula 2(c) in Section 7.2,

$\cos 2x\ \cos 6x = \frac{1}{2}[\cos(2x-6x)+\cos(2x+6x)] = \frac{1}{2}[\cos(-4x)+\cos 8x] = \frac{1}{2}(\cos 4x + \cos 8x)$. Thus,

$\int \cos 2x\ \cos 6x\,dx = \frac{1}{2}\int(\cos 4x + \cos 8x)\,dx = \frac{1}{2}\left(\frac{1}{4}\sin 4x + \frac{1}{8}\sin 8x\right) + C = \frac{1}{8}\sin 4x + \frac{1}{16}\sin 8x + C.$

36. The integrand is an odd function, so $\int_{-\pi/4}^{\pi/4} \frac{x^2\tan x}{1+\cos^4 x}\,dx = 0$ [by 4.5.6(b)].

37. Let $u = \tan\theta$. Then $du = \sec^2\theta\,d\theta \ \Rightarrow\ \int_0^{\pi/4} \tan^3\theta\ \sec^2\theta\,d\theta = \int_0^1 u^3\,du = \left[\frac{1}{4}u^4\right]_0^1 = \frac{1}{4}.$

38. $\int_{\pi/6}^{\pi/3} \frac{\sin\theta\ \cot\theta}{\sec\theta}\,d\theta = \int_{\pi/6}^{\pi/3} \cos^2\theta\,d\theta = \frac{1}{2}\int_{\pi/6}^{\pi/3}(1+\cos 2\theta)\,d\theta = \frac{1}{2}\left[\theta + \frac{1}{2}\sin 2\theta\right]_{\pi/6}^{\pi/3}$

$= \frac{1}{2}\left[\left(\frac{\pi}{3}+\frac{\sqrt{3}}{4}\right) - \left(\frac{\pi}{6}+\frac{\sqrt{3}}{4}\right)\right] = \frac{1}{2}\left(\frac{\pi}{6}\right) = \frac{\pi}{12}$

39. Let $u = \sec\theta$, so that $du = \sec\theta\,\tan\theta\,d\theta$. Then $\displaystyle\int \frac{\sec\theta\,\tan\theta}{\sec^2\theta - \sec\theta}\,d\theta = \int \frac{1}{u^2 - u}\,du = \int \frac{1}{u(u-1)}\,du = I$. Now

$\dfrac{1}{u(u-1)} = \dfrac{A}{u} + \dfrac{B}{u-1} \quad\Rightarrow\quad 1 = A(u-1) + Bu$. Set $u = 1$ to get $1 = B$. Set $u = 0$ to get $1 = -A$, so $A = -1$.

Thus, $I = \displaystyle\int \left(\frac{-1}{u} + \frac{1}{u-1}\right) du = -\ln|u| + \ln|u-1| + C = \ln|\sec\theta - 1| - \ln|\sec\theta| + C$ [or $\ln|1 - \cos\theta| + C$].

40. $4y^2 - 4y - 3 = (2y-1)^2 - 2^2$, so let $u = 2y - 1 \quad\Rightarrow\quad du = 2\,dy$. Thus,

$$\int \frac{dy}{\sqrt{4y^2 - 4y - 3}} = \int \frac{dy}{\sqrt{(2y-1)^2 - 2^2}} = \frac{1}{2}\int \frac{du}{\sqrt{u^2 - 2^2}}$$
$$= \tfrac{1}{2}\ln\left|u + \sqrt{u^2 - 2^2}\,\right| \qquad \text{[by Formula 20 in the table in this section]}$$
$$= \tfrac{1}{2}\ln\left|2y - 1 + \sqrt{4y^2 - 4y - 3}\,\right| + C$$

41. Let $u = \theta$, $dv = \tan^2\theta\,d\theta = \left(\sec^2\theta - 1\right)d\theta \quad\Rightarrow\quad du = d\theta$ and $v = \tan\theta - \theta$. So

$$\int \theta\tan^2\theta\,d\theta = \theta(\tan\theta - \theta) - \int(\tan\theta - \theta)\,d\theta = \theta\tan\theta - \theta^2 - \ln|\sec\theta| + \tfrac{1}{2}\theta^2 + C$$
$$= \theta\tan\theta - \tfrac{1}{2}\theta^2 - \ln|\sec\theta| + C$$

42.

Let $u = \tan^{-1} x$, $dv = \frac{1}{x^2}\,dx \quad \Rightarrow \quad du = \frac{1}{1+x^2}\,dx$, $v = -\frac{1}{x}$. Then

$$I = \int \frac{\tan^{-1} x}{x^2}\,dx = -\frac{1}{x}\tan^{-1} x - \int \left(-\frac{1}{x(1+x^2)}\right) dx = -\frac{1}{x}\tan^{-1} x + \int \left(\frac{A}{x} + \frac{Bx+C}{1+x^2}\right) dx$$

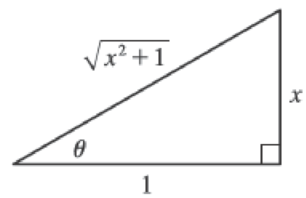
$\frac{1}{x(1+x^2)} = \frac{A}{x} + \frac{Bx+C}{1+x^2} \quad \Rightarrow \quad 1 = A(1+x^2) + (Bx+C)x \quad \Rightarrow \quad 1 = (A+B)x^2 + Cx + A$, so $C = 0$, $A = 1$,

and $A + B = 0 \quad \Rightarrow \quad B = -1$. Thus,

$$\begin{aligned} I &= -\frac{1}{x}\tan^{-1} x + \int \left(\frac{1}{x} - \frac{x}{1+x^2}\right) dx = -\frac{1}{x}\tan^{-1} x + \ln|x| - \frac{1}{2}\ln\left|1+x^2\right| + C \\ &= -\frac{\tan^{-1} x}{x} + \ln\left|\frac{x}{\sqrt{x^2+1}}\right| + C \end{aligned}$$

Or: Let $x = \tan\theta$, so that $dx = \sec^2\theta\,d\theta$. Then $\int \frac{\tan^{-1} x}{x^2}\,dx = \int \frac{\theta}{\tan^2\theta}\sec^2\theta\,d\theta = \int \theta\csc^2\theta\,d\theta = I$. Now use parts with $u = \theta$, $dv = \csc^2\theta\,d\theta \quad \Rightarrow \quad du = d\theta$, $v = -\cot\theta$. Thus,

$$\begin{aligned} I &= -\theta\cot\theta - \textstyle\int(-\cot\theta)\,d\theta = -\theta\cot\theta + \ln|\sin\theta| + C \\ &= -\tan^{-1} x \cdot \frac{1}{x} + \ln\left|\frac{x}{\sqrt{x^2+1}}\right| + C = -\frac{\tan^{-1} x}{x} + \ln\left|\frac{x}{\sqrt{x^2+1}}\right| + C \end{aligned}$$



43. Let $u = \sqrt{x}$ so that $du = \frac{1}{2\sqrt{x}}\,dx$. Then

$$\begin{aligned} \int \frac{\sqrt{x}}{1+x^3}\,dx &= \int \frac{u}{1+u^6}\,(2u\,du) = 2\int \frac{u^2}{1+(u^3)^2}\,du = 2\int \frac{1}{1+t^2}\left(\frac{1}{3}\,dt\right) \qquad \left[\begin{array}{l} t = u^3 \\ dt = 3u^2\,du \end{array}\right] \\ &= \tfrac{2}{3}\tan^{-1} t + C = \tfrac{2}{3}\tan^{-1} u^3 + C = \tfrac{2}{3}\tan^{-1}(x^{3/2}) + C \end{aligned}$$

Another method: Let $u = x^{3/2}$ so that $u^2 = x^3$ and $du = \frac{3}{2}x^{1/2}\,dx \quad \Rightarrow \quad \sqrt{x}\,dx = \frac{2}{3}\,du$. Then

$$\int \frac{\sqrt{x}}{1+x^3}\,dx = \int \frac{\frac{2}{3}}{1+u^2}\,du = \frac{2}{3}\tan^{-1} u + C = \frac{2}{3}\tan^{-1}(x^{3/2}) + C.$$

44. Let $u = \sqrt{1+e^x}$. Then $u^2 = 1 + e^x$, $2u\,du = e^x\,dx = (u^2-1)\,dx$, and $dx = \frac{2u}{u^2-1}du$, so

$$\begin{aligned} \int \sqrt{1+e^x}\,dx &= \int u \cdot \frac{2u}{u^2-1}\,du = \int \frac{2u^2}{u^2-1}\,du = \int \left(2 + \frac{2}{u^2-1}\right) du = \int \left(2 + \frac{1}{u-1} - \frac{1}{u+1}\right) du \\ &= 2u + \ln|u-1| - \ln|u+1| + C = 2\sqrt{1+e^x} + \ln\left(\sqrt{1+e^x} - 1\right) - \ln\left(\sqrt{1+e^x} + 1\right) + C \end{aligned}$$

45. Let $t = x^3$. Then $dt = 3x^2\,dx \quad \Rightarrow \quad I = \int x^5 e^{-x^3}\,dx = \frac{1}{3}\int te^{-t}\,dt$. Now integrate by parts with $u = t$, $dv = e^{-t}\,dt$:

$I = -\frac{1}{3}te^{-t} + \frac{1}{3}\int e^{-t}\,dt = -\frac{1}{3}te^{-t} - \frac{1}{3}e^{-t} + C = -\frac{1}{3}e^{-x^3}(x^3+1) + C.$

46. Use integration by parts with $u = (x-1)e^x$, $dv = \dfrac{1}{x^2}\,dx \quad \Rightarrow \quad du = [(x-1)e^x + e^x]\,dx = xe^x\,dx$, $v = -\dfrac{1}{x}$. Then

$$\int \frac{(x-1)e^x}{x^2}\,dx = (x-1)e^x\left(-\frac{1}{x}\right) - \int -e^x\,dx = -e^x + \frac{e^x}{x} + e^x + C = \frac{e^x}{x} + C.$$

47. Let $u = x - 1$, so that $du = dx$. Then

$$\begin{aligned}\int x^3(x-1)^{-4}\,dx &= \int (u+1)^3u^{-4}\,du = \int (u^3+3u^2+3u+1)u^{-4}\,du = \int (u^{-1}+3u^{-2}+3u^{-3}+u^{-4})\,du \\ &= \ln|u| - 3u^{-1} - \tfrac{3}{2}u^{-2} - \tfrac{1}{3}u^{-3} + C = \ln|x-1| - 3(x-1)^{-1} - \tfrac{3}{2}(x-1)^{-2} - \tfrac{1}{3}(x-1)^{-3} + C\end{aligned}$$

48. Let $u = \sqrt{1-x^2}$, so $u^2 = 1 - x^2$, and $2u\,du = -2x\,dx$. Then $\int_0^1 x\sqrt{2-\sqrt{1-x^2}}\,dx = \int_1^0 \sqrt{2-u}\,(-u\,du)$.

Now let $v = \sqrt{2-u}$, so $v^2 = 2 - u$, and $2v\,dv = -du$. Thus,

$$\begin{aligned}\int_1^0 \sqrt{2-u}\,(-u\,du) &= \int_1^{\sqrt{2}} v(2-v^2)\,(2v\,dv) = \int_1^{\sqrt{2}} (4v^2 - 2v^4)\,dv = \left[\tfrac{4}{3}v^3 - \tfrac{2}{5}v^5\right]_1^{\sqrt{2}} \\ &= \left(\tfrac{8}{3}\sqrt{2} - \tfrac{8}{5}\sqrt{2}\right) - \left(\tfrac{4}{3} - \tfrac{2}{5}\right) = \tfrac{16}{15}\sqrt{2} - \tfrac{14}{15}\end{aligned}$$

49. Let $u = \sqrt{4x+1} \quad \Rightarrow \quad u^2 = 4x + 1 \quad \Rightarrow \quad 2u\,du = 4\,dx \quad \Rightarrow \quad dx = \frac{1}{2}u\,du$. So

$$\begin{aligned}\int \frac{1}{x\sqrt{4x+1}}\,dx &= \int \frac{\frac{1}{2}u\,du}{\frac{1}{4}(u^2-1)\,u} = 2\int \frac{du}{u^2-1} = 2\left(\tfrac{1}{2}\right)\ln\left|\frac{u-1}{u+1}\right| + C \qquad \text{[by Formula 19]} \\ &= \ln\left|\frac{\sqrt{4x+1}-1}{\sqrt{4x+1}+1}\right| + C\end{aligned}$$

50. As in Exercise 49, let $u = \sqrt{4x+1}$. Then $\int \frac{dx}{x^2\sqrt{4x+1}} = \int \frac{\frac{1}{2}u\,du}{\left[\frac{1}{4}(u^2-1)\right]^2 u} = 8\int \frac{du}{(u^2-1)^2}$. Now

$\frac{1}{(u^2-1)^2} = \frac{1}{(u+1)^2(u-1)^2} = \frac{A}{u+1} + \frac{B}{(u+1)^2} + \frac{C}{u-1} + \frac{D}{(u-1)^2} \quad \Rightarrow$

$1 = A(u+1)(u-1)^2 + B(u-1)^2 + C(u-1)(u+1)^2 + D(u+1)^2$. $u = 1 \quad \Rightarrow \quad D = \frac{1}{4}$, $u = -1 \quad \Rightarrow \quad B = \frac{1}{4}$.

Equating coefficients of u^3 gives $A + C = 0$, and equating coefficients of 1 gives $1 = A + B - C + D \quad \Rightarrow$

$1 = A + \frac{1}{4} - C + \frac{1}{4} \quad \Rightarrow \quad \frac{1}{2} = A - C$. So $A = \frac{1}{4}$ and $C = -\frac{1}{4}$. Therefore,

$$\begin{aligned}
\int \frac{dx}{x^2\sqrt{4x+1}} &= 8\int \left[\frac{1/4}{u+1} + \frac{1/4}{(u+1)^2} + \frac{-1/4}{u-1} + \frac{1/4}{(u-1)^2}\right] du \\
&= \int \left[\frac{2}{u+1} + 2(u+1)^{-2} - \frac{2}{u-1} + 2(u-1)^{-2}\right] du \\
&= 2\ln|u+1| - \frac{2}{u+1} - 2\ln|u-1| - \frac{2}{u-1} + C \\
&= 2\ln\left(\sqrt{4x+1}+1\right) - \frac{2}{\sqrt{4x+1}+1} - 2\ln\left|\sqrt{4x+1}-1\right| - \frac{2}{\sqrt{4x+1}-1} + C
\end{aligned}$$