1. (a) $\dfrac{1+6x}{(4x-3)(2x+5)} = \dfrac{A}{4x-3} + \dfrac{B}{2x+5}$

(b) $\dfrac{10}{5x^2-2x^3} = \dfrac{10}{x^2(5-2x)} = \dfrac{A}{x} + \dfrac{B}{x^2} + \dfrac{C}{5-2x}$

2. (a) $\dfrac{x}{x^2+x-2} = \dfrac{x}{(x+2)(x-1)} = \dfrac{A}{x+2} + \dfrac{B}{x-1}$

(b) $\dfrac{x^2}{x^2+x+2} = \dfrac{(x^2+x+2)-(x+2)}{x^2+x+2} = 1 - \dfrac{x+2}{x^2+x+2}$

Notice that x^2+x+2 can't be factored because its discriminant is $b^2-4ac=-7<0$.

3. (a) $\dfrac{x^4+1}{x^5+4x^3} = \dfrac{x^4+1}{x^3(x^2+4)} = \dfrac{A}{x} + \dfrac{B}{x^2} + \dfrac{C}{x^3} + \dfrac{Dx+E}{x^2+4}$

(b) $\dfrac{1}{(x^2-9)^2} = \dfrac{1}{[(x+3)(x-3)]^2} = \dfrac{1}{(x+3)^2(x-3)^2} = \dfrac{A}{x+3} + \dfrac{B}{(x+3)^2} + \dfrac{C}{x-3} + \dfrac{D}{(x-3)^2}$

4. (a) $\dfrac{x^4-2x^3+x^2+2x-1}{x^2-2x+1} = \dfrac{x^2(x^2-2x+1)+2x-1}{x^2-2x+1} = x^2 + \dfrac{2x-1}{(x-1)^2}$ [or use long division]

$$= x^2 + \frac{A}{x-1} + \frac{B}{(x-1)^2}$$

(b) $\dfrac{x^2-1}{x^3+x^2+x} = \dfrac{x^2-1}{x(x^2+x+1)} = \dfrac{A}{x} + \dfrac{Bx+C}{x^2+x+1}$

5. (a) $\dfrac{x^6}{x^2-4} = x^4+4x^2+16+\dfrac{64}{(x+2)(x-2)}$ [by long division]

$$= x^4+4x^2+16+\frac{A}{x+2}+\frac{B}{x-2}$$

(b) $\dfrac{x^4}{(x^2-x+1)(x^2+2)^2} = \dfrac{Ax+B}{x^2-x+1} + \dfrac{Cx+D}{x^2+2} + \dfrac{Ex+F}{(x^2+2)^2}$

6.

(a) $\dfrac{t^6+1}{t^6+t^3} = \dfrac{(t^6+t^3)-t^3+1}{t^6+t^3} = 1+\dfrac{-t^3+1}{t^3(t^3+1)} = 1+\dfrac{-t^3+1}{t^3(t+1)(t^2-t+1)} = 1+\dfrac{A}{t}+\dfrac{B}{t^2}+\dfrac{C}{t^3}+\dfrac{D}{t+1}+\dfrac{Ex+F}{t^2-t+1}$

(b) $\dfrac{x^5+1}{(x^2-x)(x^4+2x^2+1)} = \dfrac{x^5+1}{x(x-1)(x^2+1)^2} = \dfrac{A}{x} + \dfrac{B}{x-1} + \dfrac{Cx+D}{x^2+1} + \dfrac{Ex+F}{(x^2+1)^2}$

7. $\displaystyle\int \frac{x^4}{x-1}\,dx = \int \left(x^3+x^2+x+1+\frac{1}{x-1}\right)dx$ [by division] $= \dfrac{1}{4}x^4+\dfrac{1}{3}x^3+\dfrac{1}{2}x^2+x+\ln|x-1|+C$

8. $\displaystyle\int \frac{3t-2}{t+1}\,dt = \int \left(3-\frac{5}{t+1}\right)dt = 3t-5\ln|t+1|+C$

9.

$\dfrac{5x+1}{(2x+1)(x-1)} = \dfrac{A}{2x+1} + \dfrac{B}{x-1}$. Multiply both sides by $(2x+1)(x-1)$ to get $5x+1 = A(x-1) + B(2x+1) \quad \Rightarrow$

$5x+1 = Ax - A + 2Bx + B \quad \Rightarrow \quad 5x+1 = (A+2B)x + (-A+B)$.

The coefficients of x must be equal and the constant terms are also equal, so $A + 2B = 5$ and

$-A + B = 1$. Adding these equations gives us $3B = 6 \quad \Leftrightarrow \quad B = 2$, and hence, $A = 1$. Thus,

$$\int \frac{5x+1}{(2x+1)(x-1)}\,dx = \int \left(\frac{1}{2x+1} + \frac{2}{x-1} \right) dx = \tfrac{1}{2}\ln|2x+1| + 2\ln|x-1| + C.$$

Another method: Substituting 1 for x in the equation $5x+1 = A(x-1) + B(2x+1)$ gives $6 = 3B \quad \Leftrightarrow \quad B = 2$.

Substituting $-\frac{1}{2}$ for x gives $-\frac{3}{2} = -\frac{3}{2}A \quad \Leftrightarrow \quad A = 1$.

10.

$\dfrac{y}{(y+4)(2y-1)} = \dfrac{A}{y+4} + \dfrac{B}{2y-1}$. Multiply both sides by $(y+4)(2y-1)$ to get $y = A(2y-1) + B(y+4) \quad \Rightarrow$

$y = 2Ay - A + By + 4B \quad \Rightarrow \quad y = (2A+B)y + (-A+4B)$. The coefficients of y must be equal and the constant terms are also equal, so $2A + B = 1$ and $-A + 4B = 0$. Adding 2 times the second equation and the first equation gives us

$9B = 1 \quad \Leftrightarrow \quad B = \frac{1}{9}$ and hence, $A = \frac{4}{9}$. Thus,

$$\begin{aligned}\int \frac{y\,dy}{(y+4)(2y-1)} &= \int \left(\frac{\frac{4}{9}}{y+4} + \frac{\frac{1}{9}}{2y-1} \right) dy = \frac{4}{9}\ln|y+4| + \frac{1}{9}\cdot\frac{1}{2}\ln|2y-1| + C \\ &= \tfrac{4}{9}\ln|y+4| + \tfrac{1}{18}\ln|2y-1| + C\end{aligned}$$

Another method: Substituting $\frac{1}{2}$ for y in the equation $y = A(2y-1) + B(y+4)$ gives $\frac{1}{2} = \frac{9}{2}B \quad \Leftrightarrow \quad B = \frac{1}{9}$.

Substituting -4 for y gives $-4 = -9A \quad \Leftrightarrow \quad A = \frac{4}{9}$.

11.

$\dfrac{2}{2x^2+3x+1} = \dfrac{2}{(2x+1)(x+1)} = \dfrac{A}{2x+1} + \dfrac{B}{x+1}$. Multiply both sides by $(2x+1)(x+1)$ to get

$2 = A(x+1) + B(2x+1)$. The coefficients of x must be equal and the constant terms are also equal, so $A + 2B = 0$ and

$A + B = 2$. Subtracting the second equation from the first gives $B = -2$, and hence, $A = 4$. Thus,

$$\int_0^1 \frac{2}{2x^2+3x+1}\,dx = \int_0^1 \left(\frac{4}{2x+1} - \frac{2}{x+1} \right) dx = \left[\frac{4}{2}\ln|2x+1| - 2\ln|x+1| \right]_0^1 = (2\ln 3 - 2\ln 2) - 0 = 2\ln\frac{3}{2}.$$

Another method: Substituting -1 for x in the equation $2 = A(x+1) + B(2x+1)$ gives $2 = -B \quad \Leftrightarrow \quad B = -2$.

Substituting $-\frac{1}{2}$ for x gives $2 = \frac{1}{2}A \quad \Leftrightarrow \quad A = 4$.

12. $\dfrac{x-4}{x^2-5x+6} = \dfrac{A}{x-2} + \dfrac{B}{x-3}$. Multiply both sides by $(x-2)(x-3)$ to get $x-4 = A(x-3)+B(x-2) \quad \Rightarrow$

$x-4 = Ax-3A+Bx-2B \quad \Rightarrow \quad x-4 = (A+B)x + (-3A-2B)$.

The coefficients of x must be equal and the constant terms are also equal, so $A+B=1$ and $-3A-2B=-4$.

Adding twice the first equation to the second gives us $-A=-2 \quad \Leftrightarrow \quad A=2$, and hence, $B=-1$.Thus,

$$\int_0^1 \frac{x-4}{x^2-5x+6}\,dx = \int_0^1 \left(\frac{2}{x-2} - \frac{1}{x-3}\right) dx = \left[2\ln|x-2| - \ln|x-3|\right]_0^1$$

$$= (0-\ln 2) - (2\ln 2 - \ln 3) = -3\ln 2 + \ln 3 \;\; [\text{or } \ln \tfrac{3}{8}]$$

Another method: Substituting 3 for x in the equation $x-4 = A(x-3)+B(x-2)$ gives $-1 = B$. Substituting 2 for x gives $-2 = -A \quad \Leftrightarrow \quad A = 2$.

13. $\displaystyle\int \frac{ax}{x^2-bx}\,dx = \int \frac{ax}{x(x-b)}\,dx = \int \frac{a}{x-b}\,dx = a\ln|x-b| + C$

14. If $a \neq b$, $\dfrac{1}{(x+a)(x+b)} = \dfrac{1}{b-a}\left(\dfrac{1}{x+a} - \dfrac{1}{x+b}\right)$, so if $a \neq b$, then

$$\int \frac{dx}{(x+a)(x+b)} = \frac{1}{b-a}\left(\ln|x+a| - \ln|x+b|\right) + C = \frac{1}{b-a}\ln\left|\frac{x+a}{x+b}\right| + C$$

If $a = b$, then $\displaystyle\int \frac{dx}{(x+a)^2} = -\frac{1}{x+a} + C$.

15. $\dfrac{x^3-2x^2-4}{x^3-2x^2} = 1 + \dfrac{-4}{x^2(x-2)}$. Write $\dfrac{-4}{x^2(x-2)} = \dfrac{A}{x} + \dfrac{B}{x^2} + \dfrac{C}{x-2}$. Multiplying both sides by $x^2(x-2)$ gives $-4 = Ax(x-2) + B(x-2) + Cx^2$. Substituting 0 for x gives $-4 = -2B \quad \Leftrightarrow \quad B = 2$. Substituting 2 for x gives $-4 = 4C \quad \Leftrightarrow \quad C = -1$. Equating coefficients of x^2, we get $0 = A + C$, so $A = 1$. Thus,

$$\int_3^4 \frac{x^3-2x^2-4}{x^3-2x^2}\,dx = \int_3^4 \left(1 + \frac{1}{x} + \frac{2}{x^2} - \frac{1}{x-2}\right) dx = \left[x + \ln|x| - \frac{2}{x} - \ln|x-2|\right]_3^4$$

$$= \left[\left(4 + \ln 4 - \tfrac{1}{2} - \ln 2\right) - \left(3 + \ln 3 - \tfrac{2}{3} - 0\right)\right] = \tfrac{7}{6} + \ln\tfrac{2}{3}$$

16.

$\dfrac{x^3-4x-10}{x^2-x-6}=x+1+\dfrac{3x-4}{(x-3)(x+2)}$. Write $\dfrac{3x-4}{(x-3)(x+2)}=\dfrac{A}{x-3}+\dfrac{B}{x+2}$. Then

$3x-4=A(x+2)+B(x-3)$. Taking $x=3$ and $x=-2$, we get $5=5A \quad\Leftrightarrow\quad A=1$ and $-10=-5B \quad\Leftrightarrow\quad B=2$,

so $$\int_0^1 \frac{x^3-4x-10}{x^2-x-6}\,dx=\int_0^1\left(x+1+\frac{1}{x-3}+\frac{2}{x+2}\right)dx=\left[\frac{1}{2}x^2+x+\ln|x-3|+2\ln(x+2)\right]_0^1$$
$$=\left(\tfrac{1}{2}+1+\ln 2+2\ln 3\right)-(0+0+\ln 3+2\ln 2)=\tfrac{3}{2}+\ln 3-\ln 2=\tfrac{3}{2}+\ln\tfrac{3}{2}$$

17.

$\dfrac{4y^2-7y-12}{y(y+2)(y-3)}=\dfrac{A}{y}+\dfrac{B}{y+2}+\dfrac{C}{y-3} \quad\Rightarrow\quad 4y^2-7y-12=A(y+2)(y-3)+By(y-3)+Cy(y+2)$. Setting

$y=0$ gives $-12=-6A$, so $A=2$. Setting $y=-2$ gives $18=10B$, so $B=\frac{9}{5}$. Setting $y=3$ gives $3=15C$, so $C=\frac{1}{5}$.

Now

$$\int_1^2 \frac{4y^2-7y-12}{y(y+2)(y-3)}\,dy=\int_1^2\left(\frac{2}{y}+\frac{9/5}{y+2}+\frac{1/5}{y-3}\right)dy=\left[2\ln|y|+\tfrac{9}{5}\ln|y+2|+\tfrac{1}{5}\ln|y-3|\right]_1^2$$
$$=2\ln 2+\tfrac{9}{5}\ln 4+\tfrac{1}{5}\ln 1-2\ln 1-\tfrac{9}{5}\ln 3-\tfrac{1}{5}\ln 2$$
$$=2\ln 2+\tfrac{18}{5}\ln 2-\tfrac{1}{5}\ln 2-\tfrac{9}{5}\ln 3=\tfrac{27}{5}\ln 2-\tfrac{9}{5}\ln 3=\tfrac{9}{5}(3\ln 2-\ln 3)=\tfrac{9}{5}\ln\tfrac{8}{3}$$

18. $\dfrac{x^2+2x-1}{x^3-x}=\dfrac{x^2+2x-1}{x(x+1)(x-1)}=\dfrac{A}{x}+\dfrac{B}{x+1}+\dfrac{C}{x-1}$. Multiply both sides by $x(x+1)(x-1)$ to get

$x^2+2x-1=A(x+1)(x-1)+Bx(x-1)+Cx(x+1) \quad\Rightarrow$

$x^2+2x-1=Ax^2-A+Bx^2-Bx+Cx^2+Cx \quad\Rightarrow$

$x^2+2x-1=(A+B+C)x^2+(-B+C)x-A$. Equating constant terms, we get $-A=-1 \quad\Leftrightarrow\quad A=1$.

Equating coefficients of x^2 gives $1=1+B+C \quad\Leftrightarrow\quad 0=B+C$. Equating coefficients of x gives $2=-B+C$.

Adding these equations gives $2=2C \quad\Leftrightarrow\quad C=1$, and hence, $B=-1$. Thus,

$$\int \frac{x^2+2x-1}{x^3-x}\,dx=\int\left(\frac{1}{x}-\frac{1}{x+1}+\frac{1}{x-1}\right)dx=\ln|x|-\ln|x+1|+\ln|x-1|+C=\ln\left|\frac{x(x-1)}{x+1}\right|+C.$$

Another method: Substituting 0 for x in the equation $x^2+2x-1=A(x+1)(x-1)+Bx(x-1)+Cx(x+1)$

gives $-1=-A \quad\Leftrightarrow\quad A=1$. Substituting -1 for x gives $-2=2B \quad\Leftrightarrow\quad B=-1$. Substituting 1 for x gives

$2=2C \quad\Leftrightarrow\quad C=1$.

19. $\dfrac{x^2+1}{(x-3)(x-2)^2} = \dfrac{A}{x-3} + \dfrac{B}{x-2} + \dfrac{C}{(x-2)^2}$. Multiply both sides by $(x-3)(x-2)^2$ to get

$x^2+1 = A(x-2)^2 + B(x-3)(x-2) + C(x-3)$. Setting $x=2$ gives $5=-C \Leftrightarrow C=-5$. Setting $x=3$

gives $10=A$. Equating coefficients of x^2 gives $1=A+B$, so $B=-9$. Thus,

$$\int \frac{x^2+1}{(x-3)(x-2)^2}\,dx = \int \left(\frac{10}{x-3} - \frac{9}{x-2} - \frac{5}{(x-2)^2}\right) dx = 10\ln|x-3| - 9\ln|x-2| + \frac{5}{x-2} + C$$

20.

$$\frac{x^2-5x+16}{(2x+1)(x-2)^2} = \frac{A}{2x+1} + \frac{B}{x-2} + \frac{C}{(x-2)^2} \quad\Rightarrow\quad x^2-5x+16 = A(x-2)^2 + B(x-2)(2x+1) + C(2x+1).$$

Setting $x=2$ gives $10=5C$, so $C=2$. Setting $x=-\frac{1}{2}$ gives $\frac{75}{4} = \frac{25}{4}A$, so $A=3$. Equating coefficients of x^2, we get $1=A+2B$, so $-2=2B$ and $B=-1$. Thus,

$$\int \frac{x^2-5x+16}{(2x+1)(x-2)^2}\,dx = \int \left(\frac{3}{2x+1} - \frac{1}{x-2} + \frac{2}{(x-2)^2}\right) dx = \frac{3}{2}\ln|2x+1| - \ln|x-2| - \frac{2}{x-2} + C$$

21.

$$\begin{array}{r|l} & \quad\quad\quad\quad\quad\; x \\ x^2+4 & x^3 + 0x^2 + 0x + 4 \\ & \underline{x^3 \quad\quad\quad + 4x \quad\;} \\ & \quad\quad\quad -4x + 4 \end{array}$$

By long division, $\dfrac{x^3+4}{x^2+4} = x + \dfrac{-4x+4}{x^2+4}$. Thus,

$$\int \frac{x^3+4}{x^2+4}\,dx = \int \left(x + \frac{-4x+4}{x^2+4}\right) dx = \int \left(x - \frac{4x}{x^2+4} + \frac{4}{x^2+2^2}\right) dx$$

$$= \frac{1}{2}x^2 - 4\cdot\frac{1}{2}\ln|x^2+4| + 4\cdot\frac{1}{2}\tan^{-1}\left(\frac{x}{2}\right) + C = \frac{1}{2}x^2 - 2\ln(x^2+4) + 2\tan^{-1}\left(\frac{x}{2}\right) + C$$

22. $\dfrac{1}{s^2(s-1)^2} = \dfrac{A}{s} + \dfrac{B}{s^2} + \dfrac{C}{s-1} + \dfrac{D}{(s-1)^2} \quad\Rightarrow\quad 1 = As(s-1)^2 + B(s-1)^2 + Cs^2(s-1) + Ds^2.$

Set $s=0$, giving $B=1$. Then set $s=1$ to get $D=1$. Equate the coefficients of s^3 to get

$0=A+C$ or $A=-C$, and finally set $s=2$ to get $1=2A+1-4A+4$ or $A=2$. Now

$$\int \frac{ds}{s^2(s-1)^2} = \int \left[\frac{2}{s} + \frac{1}{s^2} - \frac{2}{s-1} + \frac{1}{(s-1)^2}\right] ds = 2\ln|s| - \frac{1}{s} - 2\ln|s-1| - \frac{1}{s-1} + C.$$

23. $\dfrac{10}{(x-1)(x^2+9)} = \dfrac{A}{x-1} + \dfrac{Bx+C}{x^2+9}$. Multiply both sides by $(x-1)(x^2+9)$ to get

$10 = A(x^2+9) + (Bx+C)(x-1)$ $(\star)$. Substituting 1 for x gives $10 = 10A \quad \Leftrightarrow \quad A = 1$. Substituting 0 for x gives $10 = 9A - C \quad \Rightarrow \quad C = 9(1) - 10 = -1$. The coefficients of the x^2-terms in $(\star)$ must be equal, so $0 = A + B \quad \Rightarrow$ $B = -1$. Thus,

$$\begin{aligned}\int \frac{10}{(x-1)(x^2+9)}\,dx &= \int \left(\frac{1}{x-1} + \frac{-x-1}{x^2+9}\right)dx = \int \left(\frac{1}{x-1} - \frac{x}{x^2+9} - \frac{1}{x^2+9}\right)dx \\ &= \ln|x-1| - \tfrac{1}{2}\ln(x^2+9) - \tfrac{1}{3}\tan^{-1}\left(\tfrac{x}{3}\right) + C\end{aligned}$$

In the second term we used the substitution $u = x^2 + 9$ and in the last term we used Formula 10.

24. $\dfrac{x^2-x+6}{x^3+3x} = \dfrac{x^2-x+6}{x(x^2+3)} = \dfrac{A}{x} + \dfrac{Bx+C}{x^2+3}$. Multiply by $x(x^2+3)$ to get $x^2 - x + 6 = A(x^2+3) + (Bx+C)x$.

Substituting 0 for x gives $6 = 3A \quad \Leftrightarrow \quad A = 2$. The coefficients of the x^2-terms must be equal, so $1 = A + B \quad \Rightarrow$ $B = 1 - 2 = -1$. The coefficients of the x-terms must be equal, so $-1 = C$. Thus,

$$\begin{aligned}\int \frac{x^2-x+6}{x^3+3x}\,dx &= \int \left(\frac{2}{x} + \frac{-x-1}{x^2+3}\right)dx = \int \left(\frac{2}{x} - \frac{x}{x^2+3} - \frac{1}{x^2+3}\right)dx \\ &= 2\ln|x| - \frac{1}{2}\ln(x^2+3) - \frac{1}{\sqrt{3}}\tan^{-1}\frac{x}{\sqrt{3}} + C\end{aligned}$$

25. $\dfrac{4x}{x^3+x^2+x+1} = \dfrac{4x}{x^2(x+1)+1(x+1)} = \dfrac{4x}{(x+1)(x^2+1)} = \dfrac{A}{x+1} + \dfrac{Bx+C}{x^2+1}$. Multiply both sides by $(x+1)(x^2+1)$ to get $4x = A(x^2+1) + (Bx+C)(x+1) \quad \Leftrightarrow \quad 4x = Ax^2 + A + Bx^2 + Bx + Cx + C \quad \Leftrightarrow$ $4x = (A+B)x^2 + (B+C)x + (A+C)$. Comparing coefficients give us the following system of equations:

$$A + B = 0 \quad \textbf{(1)} \qquad B + C = 4 \quad \textbf{(2)} \qquad A + C = 0 \quad \textbf{(3)}$$

Subtracting equation **(1)** from equation **(2)** gives us $-A + C = 4$, and adding that equation to equation **(3)** gives us $2C = 4 \quad \Leftrightarrow \quad C = 2$, and hence $A = -2$ and $B = 2$. Thus,

$$\begin{aligned}\int \frac{4x}{x^3+x^2+x+1}\,dx &= \int \left(\frac{-2}{x+1} + \frac{2x+2}{x^2+1}\right)dx = \int \left(\frac{-2}{x+1} + \frac{2x}{x^2+1} + \frac{2}{x^2+1}\right)dx \\ &= -2\ln|x+1| + \ln(x^2+1) + 2\tan^{-1}x + C.\end{aligned}$$

26. $$\begin{aligned}\int \frac{x^2+x+1}{(x^2+1)^2}\,dx &= \int \frac{x^2+1}{(x^2+1)^2}\,dx + \int \frac{x}{(x^2+1)^2}\,dx = \int \frac{1}{x^2+1}\,dx + \frac{1}{2}\int \frac{1}{u^2}\,du \quad [u = x^2+1,\, du = 2x\,dx] \\ &= \tan^{-1}x + \frac{1}{2}\left(-\frac{1}{u}\right) + C = \tan^{-1}x - \frac{1}{2(x^2+1)} + C\end{aligned}$$

27.

$\dfrac{x^3+x^2+2x+1}{(x^2+1)(x^2+2)} = \dfrac{Ax+B}{x^2+1} + \dfrac{Cx+D}{x^2+2}$. Multiply both sides by $(x^2+1)(x^2+2)$ to get

$x^3+x^2+2x+1 = (Ax+B)(x^2+2) + (Cx+D)(x^2+1) \quad \Leftrightarrow$

$x^3+x^2+2x+1 = (Ax^3+Bx^2+2Ax+2B) + (Cx^3+Dx^2+Cx+D) \quad \Leftrightarrow$

$x^3+x^2+2x+1 = (A+C)x^3 + (B+D)x^2 + (2A+C)x + (2B+D)$. Comparing coefficients gives us the following system of equations:

$$\begin{aligned} A+C&=1 \quad \textbf{(1)} \qquad & B+D&=1 \quad \textbf{(2)} \\ 2A+C&=2 \quad \textbf{(3)} \qquad & 2B+D&=1 \quad \textbf{(4)} \end{aligned}$$

Subtracting equation **(1)** from equation **(3)** gives us $A=1$, so $C=\mathbf{0}$. Subtracting equation **(2)** from equation **(4)** gives us $B=0$, so $D=1$. Thus, $I=\displaystyle\int \frac{x^3+x^2+2x+1}{(x^2+1)(x^2+2)}\,dx = \int\left(\frac{x}{x^2+1}+\frac{1}{x^2+2}\right)dx$. For $\displaystyle\int \frac{x}{x^2+1}\,dx$, let $u=x^2+1$ so $du=2x\,dx$ and then $\displaystyle\int\frac{x}{x^2+1}\,dx = \frac{1}{2}\int\frac{1}{u}\,du = \frac{1}{2}\ln|u|+C = \frac{1}{2}\ln(x^2+1)+C$. For $\displaystyle\int\frac{1}{x^2+2}\,dx$, use Formula 10 with $a=\sqrt{2}$. So $\displaystyle\int\frac{1}{x^2+2}\,dx = \int\frac{1}{x^2+(\sqrt{2})^2}\,dx = \frac{1}{\sqrt{2}}\tan^{-1}\frac{x}{\sqrt{2}}+C$.

Thus, $I=\dfrac{1}{2}\ln(x^2+1)+\dfrac{1}{\sqrt{2}}\tan^{-1}\dfrac{x}{\sqrt{2}}+C$.

28.

$\dfrac{x^2-2x-1}{(x-1)^2(x^2+1)} = \dfrac{A}{x-1}+\dfrac{B}{(x-1)^2}+\dfrac{Cx+D}{x^2+1} \quad \Rightarrow$

$x^2-2x-1 = A(x-1)(x^2+1)+B(x^2+1)+(Cx+D)(x-1)^2$. Setting $x=1$ gives $B=-1$. Equating the coefficients of x^3 gives $A=-C$. Equating the constant terms gives $-1=-A-1+D$, so $D=A$, and setting $x=2$ gives $-1=5A-5-2A+A$ or $A=1$. We have

$$\int\frac{x^2-2x-1}{(x-1)^2(x^2+1)}\,dx = \int\left[\frac{1}{x-1}-\frac{1}{(x-1)^2}-\frac{x-1}{x^2+1}\right]dx = \ln|x-1|+\frac{1}{x-1}-\frac{1}{2}\ln(x^2+1)+\tan^{-1}x+C.$$

29.
$$\begin{aligned}\int\frac{x+4}{x^2+2x+5}\,dx &= \int\frac{x+1}{x^2+2x+5}\,dx+\int\frac{3}{x^2+2x+5}\,dx = \frac{1}{2}\int\frac{(2x+2)\,dx}{x^2+2x+5}+\int\frac{3\,dx}{(x+1)^2+4} \\ &= \frac{1}{2}\ln\left|x^2+2x+5\right|+3\int\frac{2\,du}{4(u^2+1)} \qquad \left[\begin{array}{c}\text{where } x+1=2u, \\ \text{and } dx=2\,du\end{array}\right] \\ &= \frac{1}{2}\ln(x^2+2x+5)+\frac{3}{2}\tan^{-1}u+C = \frac{1}{2}\ln(x^2+2x+5)+\frac{3}{2}\tan^{-1}\left(\frac{x+1}{2}\right)+C\end{aligned}$$

30.
$\frac{3x^2+x+4}{x^4+3x^2+2} = \frac{3x^2+x+4}{(x^2+1)(x^2+2)} = \frac{Ax+B}{x^2+1} + \frac{Cx+D}{x^2+2}$. Multiply both sides by $(x^2+1)(x^2+2)$ to get

$3x^2+x+4 = (Ax+B)(x^2+2)+(Cx+D)(x^2+1) \quad \Leftrightarrow$

$3x^2+x+4 = (Ax^3+Bx^2+2Ax+2B)+(Cx^3+Dx^2+Cx+D) \quad \Leftrightarrow$

$3x^2+x+4 = (A+C)x^3+(B+D)x^2+(2A+C)x+(2B+D)$. Comparing coefficients gives us the following system of equations:

$$\begin{aligned} A+C&=0 \quad \textbf{(1)} &\qquad B+D&=3 \quad \textbf{(2)} \\ 2A+C&=1 \quad \textbf{(3)} &\qquad 2B+D&=4 \quad \textbf{(4)} \end{aligned}$$

Subtracting equation **(1)** from equation **(3)** gives us $A=1$, so $C=-1$. Subtracting equation **(2)** from equation **(4)** gives us $B=1$, so $D=2$. Thus,

$$\begin{aligned} I = \int \frac{3x^2+x+4}{x^4+3x^2+2}\,dx &= \int \frac{x+1}{x^2+1}\,dx + \int \frac{-x+2}{x^2+2}\,dx \\ &= \frac{1}{2}\int \frac{2x}{x^2+1}\,dx + \int \frac{1}{x^2+1}\,dx - \frac{1}{2}\int \frac{2x}{x^2+2}\,dx + 2\int \frac{1}{x^2+(\sqrt{2})^2}\,dx \\ &= \tfrac{1}{2}\ln\left|x^2+1\right| + \tan^{-1}x - \tfrac{1}{2}\ln\left|x^2+2\right| + 2\cdot\frac{1}{\sqrt{2}}\tan^{-1}\left(\frac{x}{\sqrt{2}}\right) + C \\ &= \tfrac{1}{2}\ln(x^2+1) - \tfrac{1}{2}\ln(x^2+2) + \tan^{-1}x + \sqrt{2}\,\tan^{-1}\left(x/\sqrt{2}\right) + C \end{aligned}$$

31.
$\frac{1}{x^3-1} = \frac{1}{(x-1)(x^2+x+1)} = \frac{A}{x-1} + \frac{Bx+C}{x^2+x+1} \quad \Rightarrow \quad 1 = A\left(x^2+x+1\right)+(Bx+C)(x-1).$

Take $x=1$ to get $A=\frac{1}{3}$. Equating coefficients of x^2 and then comparing the constant terms, we get $0=\frac{1}{3}+B$, $1=\frac{1}{3}-C$, so $B=-\frac{1}{3}$, $C=-\frac{2}{3} \quad \Rightarrow$

$$\begin{aligned} \int \frac{1}{x^3-1}dx &= \int \frac{\frac{1}{3}}{x-1}\,dx + \int \frac{-\frac{1}{3}x-\frac{2}{3}}{x^2+x+1}\,dx = \tfrac{1}{3}\ln|x-1| - \frac{1}{3}\int \frac{x+2}{x^2+x+1}dx \\ &= \tfrac{1}{3}\ln|x-1| - \frac{1}{3}\int \frac{x+1/2}{x^2+x+1}dx - \frac{1}{3}\int \frac{(3/2)\,dx}{(x+1/2)^2+3/4} \\ &= \tfrac{1}{3}\ln|x-1| - \tfrac{1}{6}\ln\left(x^2+x+1\right) - \tfrac{1}{2}\left(\frac{2}{\sqrt{3}}\right)\tan^{-1}\left(\frac{x+\frac{1}{2}}{\sqrt{3}/2}\right) + K \\ &= \tfrac{1}{3}\ln|x-1| - \tfrac{1}{6}\ln(x^2+x+1) - \tfrac{1}{\sqrt{3}}\tan^{-1}\left(\tfrac{1}{\sqrt{3}}(2x+1)\right) + K \end{aligned}$$

32. $\int_0^1 \frac{x}{x^2+4x+13}\,dx = \int_0^1 \frac{\frac{1}{2}(2x+4)}{x^2+4x+13}\,dx - 2\int_0^1 \frac{dx}{(x+2)^2+9}$

$$= \frac{1}{2}\int_{13}^{18} \frac{dy}{y} - 2\int_{2/3}^{1} \frac{3\,du}{9u^2+9} \qquad \left[\begin{array}{c}\text{where } y = x^2+4x+13,\ dy = (2x+4)\,dx, \\ x+2 = 3u, \text{ and } dx = 3\,du\end{array}\right]$$

$$= \tfrac{1}{2}\left[\ln y\right]_{13}^{18} - \tfrac{2}{3}\left[\tan^{-1} u\right]_{2/3}^{1} = \tfrac{1}{2}\ln\tfrac{18}{13} - \tfrac{2}{3}\left(\tfrac{\pi}{4} - \tan^{-1}\left(\tfrac{2}{3}\right)\right)$$

$$= \tfrac{1}{2}\ln\tfrac{18}{13} - \tfrac{\pi}{6} + \tfrac{2}{3}\tan^{-1}\left(\tfrac{2}{3}\right)$$

33. Let $u = x^4 + 4x^2 + 3$, so that $du = (4x^3 + 8x)\,dx = 4(x^3 + 2x)\,dx$, $x = 0 \Rightarrow u = 3$, and $x = 1 \Rightarrow u = 8$.

Then $\int_0^1 \frac{x^3+2x}{x^4+4x^2+3}\,dx = \int_3^8 \frac{1}{u}\left(\frac{1}{4}\,du\right) = \frac{1}{4}\left[\ln|u|\right]_3^8 = \frac{1}{4}(\ln 8 - \ln 3) = \frac{1}{4}\ln\frac{8}{3}$.

34. $\frac{x^5+x-1}{x^3+1} = x^2 + \frac{-x^2+x-1}{x^3+1} = x^2 + \frac{-x^2+x-1}{(x+1)(x^2-x+1)} = x^2 + \frac{-1}{x+1}$, so

$$\int \frac{x^5+x-1}{x^3+1}\,dx = \int\left(x^2 - \frac{1}{x+1}\right)dx = \frac{1}{3}x^3 - \ln|x+1| + C$$

35. $\frac{1}{x(x^2+4)^2} = \frac{A}{x} + \frac{Bx+C}{x^2+4} + \frac{Dx+E}{(x^2+4)^2} \Rightarrow 1 = A(x^2+4)^2 + (Bx+C)x(x^2+4) + (Dx+E)x$. Setting $x = 0$ gives $1 = 16A$, so $A = \frac{1}{16}$. Now compare coefficients.

$$1 = \tfrac{1}{16}(x^4 + 8x^2 + 16) + (Bx^2 + Cx)(x^2+4) + Dx^2 + Ex$$

$$1 = \tfrac{1}{16}x^4 + \tfrac{1}{2}x^2 + 1 + Bx^4 + Cx^3 + 4Bx^2 + 4Cx + Dx^2 + Ex$$

$$1 = \left(\tfrac{1}{16} + B\right)x^4 + Cx^3 + \left(\tfrac{1}{2} + 4B + D\right)x^2 + (4C+E)x + 1$$

So $B + \frac{1}{16} = 0 \Rightarrow B = -\frac{1}{16}$, $C = 0$, $\frac{1}{2} + 4B + D = 0 \Rightarrow D = -\frac{1}{4}$, and $4C + E = 0 \Rightarrow E = 0$. Thus,

$$\int \frac{dx}{x(x^2+4)^2} = \int\left(\frac{\frac{1}{16}}{x} + \frac{-\frac{1}{16}x}{x^2+4} + \frac{-\frac{1}{4}x}{(x^2+4)^2}\right)dx = \frac{1}{16}\ln|x| - \frac{1}{16}\cdot\frac{1}{2}\ln\left|x^2+4\right| - \frac{1}{4}\left(-\frac{1}{2}\right)\frac{1}{x^2+4} + C$$

$$= \frac{1}{16}\ln|x| - \frac{1}{32}\ln(x^2+4) + \frac{1}{8(x^2+4)} + C$$

36. Let $u = x^5 + 5x^3 + 5x$, so that $du = (5x^4 + 15x^2 + 5)dx = 5(x^4 + 3x^2 + 1)dx$. Then

$$\int \frac{x^4+3x^2+1}{x^5+5x^3+5x}\,dx = \int \frac{1}{u}\left(\frac{1}{5}\,du\right) = \frac{1}{5}\ln|u| + C = \frac{1}{5}\ln\left|x^5+5x^3+5x\right| + C$$

37.

$$\frac{x^2-3x+7}{(x^2-4x+6)^2}=\frac{Ax+B}{x^2-4x+6}+\frac{Cx+D}{(x^2-4x+6)^2} \quad\Rightarrow\quad x^2-3x+7=(Ax+B)(x^2-4x+6)+Cx+D \quad\Rightarrow$$

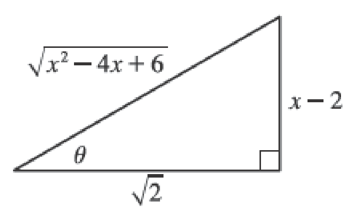
$x^2-3x+7=Ax^3+(-4A+B)x^2+(6A-4B+C)x+(6B+D)$. So $A=0$, $-4A+B=1 \;\Rightarrow\; B=1$, $6A-4B+C=-3 \;\Rightarrow\; C=1$, $6B+D=7 \;\Rightarrow\; D=1$. Thus,

$$\begin{aligned}
I&=\int\frac{x^2-3x+7}{(x^2-4x+6)^2}\,dx=\int\left(\frac{1}{x^2-4x+6}+\frac{x+1}{(x^2-4x+6)^2}\right)dx\\
&=\int\frac{1}{(x-2)^2+2}\,dx+\int\frac{x-2}{(x^2-4x+6)^2}\,dx+\int\frac{3}{(x^2-4x+6)^2}\,dx\\
&=I_1+I_2+I_3.
\end{aligned}$$

$$I_1=\int\frac{1}{(x-2)^2+\left(\sqrt{2}\,\right)^2}\,dx=\frac{1}{\sqrt{2}}\tan^{-1}\left(\frac{x-2}{\sqrt{2}}\right)+C_1$$

$$I_2=\frac{1}{2}\int\frac{2x-4}{(x^2-4x+6)^2}\,dx=\tfrac{1}{2}\int\frac{1}{u^2}\,du=\tfrac{1}{2}\left(-\frac{1}{u}\right)+C_2=-\frac{1}{2(x^2-4x+6)}+C_2$$

$$\begin{aligned}
I_3&=3\int\frac{1}{\left[(x-2)^2+\left(\sqrt{2}\,\right)^2\right]^2}\,dx=3\int\frac{1}{[2(\tan^2\theta+1)]^2}\sqrt{2}\sec^2\theta\,d\theta \quad \left[\begin{matrix}x-2=\sqrt{2}\tan\theta,\\ dx=\sqrt{2}\,\sec^2\theta\,d\theta\end{matrix}\right]\\
&=\frac{3\sqrt{2}}{4}\int\frac{\sec^2\theta}{\sec^4\theta}\,d\theta=\frac{3\sqrt{2}}{4}\int\cos^2\theta\,d\theta=\frac{3\sqrt{2}}{4}\int\tfrac{1}{2}(1+\cos 2\theta)\,d\theta\\
&=\frac{3\sqrt{2}}{8}\left(\theta+\tfrac{1}{2}\sin 2\theta\right)+C_3=\frac{3\sqrt{2}}{8}\tan^{-1}\left(\frac{x-2}{\sqrt{2}}\right)+\frac{3\sqrt{2}}{8}\left(\tfrac{1}{2}\cdot 2\sin\theta\,\cos\theta\right)+C_3\\
&=\frac{3\sqrt{2}}{8}\tan^{-1}\left(\frac{x-2}{\sqrt{2}}\right)+\frac{3\sqrt{2}}{8}\cdot\frac{x-2}{\sqrt{x^2-4x+6}}\cdot\frac{\sqrt{2}}{\sqrt{x^2-4x+6}}+C_3\\
&=\frac{3\sqrt{2}}{8}\tan^{-1}\left(\frac{x-2}{\sqrt{2}}\right)+\frac{3(x-2)}{4(x^2-4x+6)}+C_3
\end{aligned}$$



So $\quad I=I_1+I_2+I_3 \qquad [C=C_1+C_2+C_3]$

$$\begin{aligned}
&=\frac{1}{\sqrt{2}}\tan^{-1}\left(\frac{x-2}{\sqrt{2}}\right)+\frac{-1}{2(x^2-4x+6)}+\frac{3\sqrt{2}}{8}\tan^{-1}\left(\frac{x-2}{\sqrt{2}}\right)+\frac{3(x-2)}{4(x^2-4x+6)}+C\\
&=\left(\frac{4\sqrt{2}}{8}+\frac{3\sqrt{2}}{8}\right)\tan^{-1}\left(\frac{x-2}{\sqrt{2}}\right)+\frac{3(x-2)-2}{4(x^2-4x+6)}+C=\frac{7\sqrt{2}}{8}\tan^{-1}\left(\frac{x-2}{\sqrt{2}}\right)+\frac{3x-8}{4(x^2-4x+6)}+C
\end{aligned}$$

38. $\dfrac{x^3+2x^2+3x-2}{(x^2+2x+2)^2} = \dfrac{Ax+B}{x^2+2x+2} + \dfrac{Cx+D}{(x^2+2x+2)^2} \quad \Rightarrow$

$x^3+2x^2+3x-2 = (Ax+B)(x^2+2x+2)+Cx+D \quad \Rightarrow$

$x^3+2x^2+3x-2 = Ax^3+(2A+B)x^2+(2A+2B+C)x+2B+D.$

So $A=1$, $2A+B=2 \quad \Rightarrow \quad B=0$, $2A+2B+C=3 \quad \Rightarrow \quad C=1$, and $2B+D=-2 \quad \Rightarrow \quad D=-2$. Thus,

$$I = \int \frac{x^3+2x^2+3x-2}{(x^2+2x+2)^2}\,dx = \int \left(\frac{x}{x^2+2x+2} + \frac{x-2}{(x^2+2x+2)^2}\right)dx$$

$$= \int \frac{x+1}{x^2+2x+2}\,dx + \int \frac{-1}{x^2+2x+2}\,dx + \int \frac{x+1}{(x^2+2x+2)^2}\,dx + \int \frac{-3}{(x^2+2x+2)^2}dx$$

$$= I_1+I_2+I_3+I_4.$$

$$I_1 = \int \frac{x+1}{x^2+2x+2}\,dx = \int \frac{1}{u}\left(\frac{1}{2}\,du\right) \quad \begin{bmatrix} u = x^2+2x+2, \\ du = 2(x+1)\,dx \end{bmatrix} \quad = \frac{1}{2}\ln\left|x^2+2x+2\right| + C_1$$

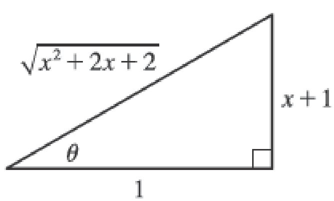
$$I_2 = -\int \frac{1}{(x+1)^2+1}\,dx = -\frac{1}{1}\tan^{-1}\left(\frac{x+1}{1}\right) + C_2 = -\tan^{-1}(x+1) + C_2$$

$$I_3 = \int \frac{x+1}{(x^2+2x+2)^2}\,dx = \int \frac{1}{u^2}\left(\frac{1}{2}\,du\right) = -\frac{1}{2u} + C_3 = -\frac{1}{2(x^2+2x+2)} + C_3$$

$$I_4 = -3\int \frac{1}{[(x+1)^2+1]^2}\,dx = -3\int \frac{1}{(\tan^2\theta+1)^2}\sec^2\theta\,d\theta \quad \begin{bmatrix} x+1 = 1\tan\theta, \\ dx = \sec^2\theta\,d\theta \end{bmatrix}$$

$$= -3\int \frac{1}{\sec^2\theta}\,d\theta = -3\int \cos^2\theta\,d\theta = -\frac{3}{2}\int (1+\cos 2\theta)\,d\theta$$

$$= -\tfrac{3}{2}\left(\theta + \tfrac{1}{2}\sin 2\theta\right) + C_4 = -\tfrac{3}{2}\theta - \tfrac{3}{2}\left(\tfrac{1}{2}\cdot 2\sin\theta\,\cos\theta\right) + C_4$$



$$= -\frac{3}{2}\tan^{-1}\left(\frac{x+1}{1}\right) - \frac{3}{2}\cdot\frac{x+1}{\sqrt{x^2+2x+2}}\cdot\frac{1}{\sqrt{x^2+2x+2}} + C_4$$

$$= -\frac{3}{2}\tan^{-1}(x+1) - \frac{3(x+1)}{2(x^2+2x+2)} + C_4$$

So $\quad I = I_1+I_2+I_3+I_4 \qquad [C = C_1+C_2+C_3+C_4]$

$$= \frac{1}{2}\ln(x^2+2x+2) - \tan^{-1}(x+1) - \frac{1}{2(x^2+2x+2)} - \frac{3}{2}\tan^{-1}(x+1) - \frac{3(x+1)}{2(x^2+2x+2)} + C$$

$$= \frac{1}{2}\ln(x^2+2x+2) - \frac{5}{2}\tan^{-1}(x+1) - \frac{3x+4}{2(x^2+2x+2)} + C$$

39. Let $u = \sqrt{x+1}$, so $u^2 = x+1$ and $2u\,du = dx$. Then

$$\int \frac{\sqrt{x+1}}{x}\,dx = \int \frac{u}{u^2-1}\,(2u\,du) = \int \frac{2u^2}{u^2-1}\,du = \int \left(2 + \frac{2}{u^2-1}\right) du.$$

$$\frac{2}{u^2-1} = \frac{2}{(u+1)(u-1)} = \frac{A}{u+1} + \frac{B}{u-1} \quad \Rightarrow \quad 2 = A(u-1) + B(u+1).$$ Setting $u=1$ gives $B=1$.

Setting $u=-1$ gives $A=-1$. Thus,

$$\int \left(2 + \frac{2}{u^2-1}\right) du = \int \left(2 - \frac{1}{u+1} + \frac{1}{u-1}\right) du = 2u - \ln|u+1| + \ln|u-1| + C$$
$$= 2\sqrt{x+1} - \ln\left(\sqrt{x+1}+1\right) + \ln\left|\sqrt{x+1}-1\right| + C.$$

40. Let $u = \sqrt{x+3}$, so $u^2 = x+3$ and $2u\,du = dx$. Then

$$\int \frac{dx}{2\sqrt{x+3}+x} = \int \frac{2u\,du}{2u+(u^2-3)} = \int \frac{2u}{u^2+2u-3}\,du = \int \frac{2u}{(u+3)(u-1)}\,du.$$ Now

$$\frac{2u}{(u+3)(u-1)} = \frac{A}{u+3} + \frac{B}{u-1} \quad \Rightarrow \quad 2u = A(u-1) + B(u+3).$$ Setting $u=1$ gives $2 = 4B$, so $B = \frac{1}{2}$.

Setting $u=-3$ gives $-6 = -4A$, so $A = \frac{3}{2}$. Thus,

$$\int \frac{2u}{(u+3)(u-1)}\,du = \int \left(\frac{\frac{3}{2}}{u+3} + \frac{\frac{1}{2}}{u-1}\,du\right)$$
$$= \tfrac{3}{2}\ln|u+3| + \tfrac{1}{2}\ln|u-1| + C = \tfrac{3}{2}\ln\left(\sqrt{x+3}+3\right) + \tfrac{1}{2}\ln\left|\sqrt{x+3}-1\right| + C$$

41.

Let $u = \sqrt{x}$, so $u^2 = x$ and $2u\,du = dx$. Then $\displaystyle\int \frac{dx}{x^2+x\sqrt{x}} = \int \frac{2u\,du}{u^4+u^3} = \int \frac{2\,du}{u^3+u^2} = \int \frac{2\,du}{u^2(u+1)}$.

$$\frac{2}{u^2(u+1)} = \frac{A}{u} + \frac{B}{u^2} + \frac{C}{u+1} \quad \Rightarrow \quad 2 = Au(u+1) + B(u+1) + Cu^2.$$ Setting $u=0$ gives $B=2$. Setting $u=-1$

gives $C=2$. Equating coefficients of u^2, we get $0 = A + C$, so $A = -2$. Thus,

$$\int \frac{2\,du}{u^2(u+1)} = \int \left(\frac{-2}{u} + \frac{2}{u^2} + \frac{2}{u+1}\right) du = -2\ln|u| - \frac{2}{u} + 2\ln|u+1| + C = -2\ln\sqrt{x} - \frac{2}{\sqrt{x}} + 2\ln\left(\sqrt{x}+1\right) + C.$$

42. Let $u = \sqrt[3]{x}$. Then $x = u^3$, $dx = 3u^2\,du \quad \Rightarrow$

$$\int_0^1 \frac{1}{1+\sqrt[3]{x}}\,dx = \int_0^1 \frac{3u^2\,du}{1+u} = \int_0^1 \left(3u - 3 + \frac{3}{1+u}\right) du = \left[\tfrac{3}{2}u^2 - 3u + 3\ln(1+u)\right]_0^1 = 3\left(\ln 2 - \tfrac{1}{2}\right).$$

43. Let $u = \sqrt[3]{x^2+1}$. Then $x^2 = u^3 - 1$, $2x\,dx = 3u^2\,du \quad \Rightarrow$

$$\int \frac{x^3\,dx}{\sqrt[3]{x^2+1}} = \int \frac{(u^3-1)\frac{3}{2}u^2\,du}{u} = \frac{3}{2}\int (u^4 - u)\,du$$
$$= \tfrac{3}{10}u^5 - \tfrac{3}{4}u^2 + C = \tfrac{3}{10}(x^2+1)^{5/3} - \tfrac{3}{4}(x^2+1)^{2/3} + C$$

44. Let $u = \sqrt{x}$. Then $x = u^2$, $dx = 2u\,du \quad \Rightarrow$

$$\int_{1/3}^{3} \frac{\sqrt{x}}{x^2 + x}\,dx = \int_{1/\sqrt{3}}^{\sqrt{3}} \frac{u \cdot 2u\,du}{u^4 + u^2} = 2\int_{1/\sqrt{3}}^{\sqrt{3}} \frac{du}{u^2+1} = 2\big[\tan^{-1} u\big]_{1/\sqrt{3}}^{\sqrt{3}} = 2\left(\tfrac{\pi}{3} - \tfrac{\pi}{6}\right) = \tfrac{\pi}{3}.$$

45.

If we were to substitute $u = \sqrt{x}$, then the square root would disappear but a cube root would remain. On the other hand, the substitution $u = \sqrt[3]{x}$ would eliminate the cube root but leave a square root. We can eliminate both roots by means of the substitution $u = \sqrt[6]{x}$. (Note that 6 is the least common multiple of 2 and 3.)

Let $u = \sqrt[6]{x}$. Then $x = u^6$, so $dx = 6u^5\,du$ and $\sqrt{x} = u^3$, $\sqrt[3]{x} = u^2$. Thus,

$$\int \frac{dx}{\sqrt{x} - \sqrt[3]{x}} = \int \frac{6u^5\,du}{u^3 - u^2} = 6\int \frac{u^5}{u^2(u-1)}\,du = 6\int \frac{u^3}{u-1}\,du$$

$$= 6\int \left(u^2 + u + 1 + \frac{1}{u-1}\right) du \qquad \text{[by long division]}$$

$$= 6\left(\tfrac{1}{3}u^3 + \tfrac{1}{2}u^2 + u + \ln|u-1|\right) + C = 2\sqrt{x} + 3\sqrt[3]{x} + 6\sqrt[6]{x} + 6\ln\left|\sqrt[6]{x} - 1\right| + C$$

46.

Let $u = \sqrt{1+\sqrt{x}}$, so that $u^2 = 1 + \sqrt{x}$, $x = (u^2-1)^2$, and $dx = 2(u^2-1)\cdot 2u\,du = 4u(u^2-1)\,du$. Then

$$\int \frac{\sqrt{1+\sqrt{x}}}{x}\,dx = \int \frac{u}{(u^2-1)^2}\cdot 4u(u^2-1)\,du = \int \frac{4u^2}{u^2-1}\,du = \int \left(4 + \frac{4}{u^2-1}\right) du.$$ Now

$\dfrac{4}{u^2-1} = \dfrac{A}{u+1} + \dfrac{B}{u-1} \quad \Rightarrow \quad 4 = A(u-1) + B(u+1)$. Setting $u = 1$ gives $4 = 2B$, so $B = 2$. Setting $u = -1$ gives $4 = -2A$, so $A = -2$. Thus,

$$\int \left(4 + \frac{4}{u^2-1}\right) du = \int \left(4 - \frac{2}{u+1} + \frac{2}{u-1}\right) du = 4u - 2\ln|u+1| + 2\ln|u-1| + C$$

$$= 4\sqrt{1+\sqrt{x}} - 2\ln\left(\sqrt{1+\sqrt{x}} + 1\right) + 2\ln\left(\sqrt{1+\sqrt{x}} - 1\right) + C$$

47. Let $u = e^x$. Then $x = \ln u$, $dx = \dfrac{du}{u} \quad \Rightarrow$

$$\int \frac{e^{2x}\,dx}{e^{2x} + 3e^x + 2} = \int \frac{u^2\,(du/u)}{u^2 + 3u + 2} = \int \frac{u\,du}{(u+1)(u+2)} = \int \left[\frac{-1}{u+1} + \frac{2}{u+2}\right] du$$

$$= 2\ln|u+2| - \ln|u+1| + C = \ln\frac{(e^x+2)^2}{e^x+1} + C$$

48. Let $u = \cos x$, so that $du = -\sin x\,dx$. Then $\displaystyle\int \frac{\sin x}{\cos^2 x - 3\cos x}\,dx = \int \frac{1}{u^2 - 3u}\,(-du) = \int \frac{-1}{u(u-3)}\,du$.

$\displaystyle\frac{-1}{u(u-3)} = \frac{A}{u} + \frac{B}{u-3} \quad\Rightarrow\quad -1 = A(u-3) + Bu$. Setting $u = 3$ gives $B = -\frac{1}{3}$. Setting $u = 0$ gives $A = \frac{1}{3}$.

Thus, $\displaystyle\int \frac{-1}{u(u-3)}\,du = \int \left(\frac{\frac{1}{3}}{u} - \frac{\frac{1}{3}}{u-3}\right) du = \frac{1}{3}\ln|u| - \frac{1}{3}\ln|u-3| + C = \tfrac{1}{3}\ln|\cos x| - \tfrac{1}{3}\ln|\cos x - 3| + C$.

49.

Let $u = \tan t$, so that $du = \sec^2 t\,dt$. Then $\displaystyle\int \frac{\sec^2 t}{\tan^2 t + 3\tan t + 2}\,dt = \int \frac{1}{u^2 + 3u + 2}\,du = \int \frac{1}{(u+1)(u+2)}\,du$.

Now $\displaystyle\frac{1}{(u+1)(u+2)} = \frac{A}{u+1} + \frac{B}{u+2} \quad\Rightarrow\quad 1 = A(u+2) + B(u+1)$.

Setting $u = -2$ gives $1 = -B$, so $B = -1$. Setting $u = -1$ gives $1 = A$.

Thus, $\displaystyle\int \frac{1}{(u+1)(u+2)}\,du = \int \left(\frac{1}{u+1} - \frac{1}{u+2}\right) du = \ln|u+1| - \ln|u+2| + C = \ln|\tan t + 1| - \ln|\tan t + 2| + C$.

50.

Let $u = e^x$, so that $du = e^x\,dx$. Then $\displaystyle\int \frac{e^x}{(e^x - 2)(e^{2x} + 1)}\,dx = \int \frac{1}{(u-2)(u^2+1)}\,du$. Now

$\displaystyle\frac{1}{(u-2)(u^2+1)} = \frac{A}{u-2} + \frac{Bu + C}{u^2+1} \quad\Rightarrow\quad 1 = A(u^2+1) + (Bu + C)(u-2)$. Setting $u = 2$ gives $1 = 5A$, so $A = \frac{1}{5}$.

Setting $u = 0$ gives $1 = \frac{1}{5} - 2C$, so $C = -\frac{2}{5}$. Comparing coefficients of u^2 gives $0 = \frac{1}{5} + B$, so $B = -\frac{1}{5}$. Thus,

$$\begin{aligned}
\int \frac{1}{(u-2)(u^2+1)}\,du &= \int \left(\frac{\frac{1}{5}}{u-2} + \frac{-\frac{1}{5}u - \frac{2}{5}}{u^2+1}\right) du = \tfrac{1}{5}\int \frac{1}{u-2}\,du - \tfrac{1}{5}\int \frac{u}{u^2+1}\,du - \tfrac{2}{5}\int \frac{1}{u^2+1}\,du \\
&= \tfrac{1}{5}\ln|u-2| - \tfrac{1}{5}\cdot\tfrac{1}{2}\ln\left|u^2+1\right| - \tfrac{2}{5}\tan^{-1} u + C \\
&= \tfrac{1}{5}\ln|e^x - 2| - \tfrac{1}{10}\ln(e^{2x} + 1) - \tfrac{2}{5}\tan^{-1} e^x + C
\end{aligned}$$