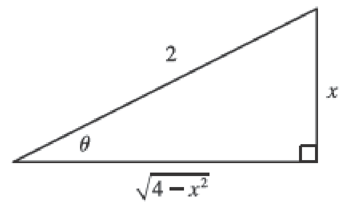
1. Let $x = 2\sin\theta$, where $-\pi/2 \le \theta \le \pi/2$. Then $dx = 2\cos\theta\, d\theta$ and

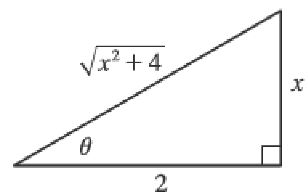


$\sqrt{4-x^2} = \sqrt{4-4\sin^2\theta} = \sqrt{4\cos^2\theta} = 2\,|\cos\theta| = 2\cos\theta$.

Thus, $\displaystyle\int \frac{dx}{x^2\sqrt{4-x^2}} = \int \frac{2\cos\theta}{4\sin^2\theta(2\cos\theta)}\,d\theta = \frac{1}{4}\int \csc^2\theta\, d\theta$

$\displaystyle = -\frac{1}{4}\cot\theta + C = -\frac{\sqrt{4-x^2}}{4x} + C$ [see figure]

2. Let $x = 2\tan\theta$, where $-\frac{\pi}{2} < \theta < \frac{\pi}{2}$. Then $dx = 2\sec^2\theta\, d\theta$ and



$\sqrt{x^2+4} = \sqrt{4\tan^2\theta + 4} = \sqrt{4(\tan^2\theta + 1)} = \sqrt{4\sec^2\theta} = 2\,|\sec\theta|$

$= 2\sec\theta$ for the relevant values of θ.

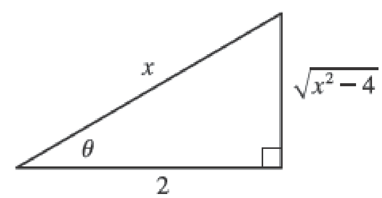
$$\int \frac{x^3}{\sqrt{x^2+4}}\,dx = \int \frac{8\tan^3\theta}{2\sec\theta}\,2\sec^2\theta\, d\theta = 8\int \tan^2\theta\,\sec\theta\,\tan\theta\, d\theta$$

$$= 8\int(\sec^2\theta - 1)\,\sec\theta\,\tan\theta\, d\theta = 8\int(u^2-1)\,du \qquad [u = \sec\theta]$$

$$= 8\left(\frac{1}{3}u^3 - u\right) + C = \frac{8}{3}\sec^3\theta - 8\sec\theta + C = \frac{8}{3}\left(\frac{\sqrt{x^2+4}}{2}\right)^3 - 8\left(\frac{\sqrt{x^2+4}}{2}\right) + C$$

$$= \tfrac{1}{3}(x^2+4)^{3/2} - 4\sqrt{x^2+4} + C$$

3. Let $x = 2\sec\theta$, where $0 \le \theta < \frac{\pi}{2}$ or $\pi \le \theta < \frac{3\pi}{2}$. Then $dx = 2\sec\theta\,\tan\theta\, d\theta$ and



$\sqrt{x^2-4} = \sqrt{4\sec^2\theta - 4} = \sqrt{4(\sec^2\theta - 1)}$

$= \sqrt{4\tan^2\theta} = 2\,|\tan\theta| = 2\tan\theta$ for the relevant values of θ

$$\int \frac{\sqrt{x^2-4}}{x}\,dx = \int \frac{2\tan\theta}{2\sec\theta}\,2\sec\theta\,\tan\theta\, d\theta = 2\int \tan^2\theta\, d\theta$$

$$= 2\int(\sec^2\theta - 1)\,d\theta = 2\,(\tan\theta - \theta) + C = 2\left[\frac{\sqrt{x^2-4}}{2} - \sec^{-1}\left(\frac{x}{2}\right)\right] + C$$

$$= \sqrt{x^2-4} - 2\sec^{-1}\left(\frac{x}{2}\right) + C$$

4. Let $x = \sin\theta$, so $dx = \cos\theta\, d\theta$, $x = 0 \;\Rightarrow\; \theta = 0$, and $x = 1 \;\Rightarrow\; \theta = \frac{\pi}{2}$. Then

$$I = \int_0^1 x^3\sqrt{1-x^2}\,dx = \int_0^{\pi/2}\sin^3\theta\,\sqrt{\cos^2\theta}\,\cos\theta\, d\theta = \int_0^{\pi/2}\sin^2\theta\,\sin\theta\,\cos^2\theta\, d\theta$$

$$= \int_0^{\pi/2}(1-\cos^2\theta)\,\sin\theta\,\cos^2\theta\, d\theta \overset{c}{=} \int_1^0(1-u^2)u^2\,(-du) = \int_0^1(u^2-u^4)\,du$$

$$= \left[\tfrac{1}{3}u^3 - \tfrac{1}{5}u^5\right]_0^1 = \tfrac{1}{3} - \tfrac{1}{5} = \tfrac{5-3}{15} = \tfrac{2}{15}$$

Another method: Let $u = 1 - x^2$, so $du = -2x\,dx$. Then

$$I = \int_1^0(1-u)\sqrt{u}\left(-\tfrac{1}{2}du\right) = \tfrac{1}{2}\int_0^1(u^{1/2} - u^{3/2})\,du = \tfrac{1}{2}\left[\tfrac{2}{3}u^{3/2} - \tfrac{2}{5}u^{5/2}\right]_0^1 = \tfrac{1}{2}\left(\tfrac{2}{3} - \tfrac{2}{5}\right) = \tfrac{1}{2}\left(\tfrac{4}{15}\right) = \tfrac{2}{15}.$$

5. Let $t = \sec\theta$, so $dt = \sec\theta\,\tan\theta\,d\theta$, $t = \sqrt{2} \;\Rightarrow\; \theta = \frac{\pi}{4}$, and $t = 2 \;\Rightarrow\; \theta = \frac{\pi}{3}$. Then

$$\int_{\sqrt{2}}^{2} \frac{1}{t^3\sqrt{t^2-1}}\,dt = \int_{\pi/4}^{\pi/3} \frac{1}{\sec^3\theta\,\tan\theta}\sec\theta\,\tan\theta\,d\theta = \int_{\pi/4}^{\pi/3} \frac{1}{\sec^2\theta}\,d\theta = \int_{\pi/4}^{\pi/3} \cos^2\theta\,d\theta$$

$$= \int_{\pi/4}^{\pi/3} \tfrac{1}{2}(1+\cos 2\theta)\,d\theta = \tfrac{1}{2}\left[\theta + \tfrac{1}{2}\sin 2\theta\right]_{\pi/4}^{\pi/3}$$

$$= \tfrac{1}{2}\left[\left(\tfrac{\pi}{3} + \tfrac{1}{2}\tfrac{\sqrt{3}}{2}\right) - \left(\tfrac{\pi}{4} + \tfrac{1}{2}\cdot 1\right)\right] = \tfrac{1}{2}\left(\tfrac{\pi}{12} + \tfrac{\sqrt{3}}{4} - \tfrac{1}{2}\right) = \tfrac{\pi}{24} + \tfrac{\sqrt{3}}{8} - \tfrac{1}{4}$$

6. Let $u = 36 - x^2$, so $du = -2x\,dx$. When $x = 0$, $u = 36$; when $x = 3$, $u = 27$. Thus,

$$\int_0^3 \frac{x}{\sqrt{36-x^2}}\,dx = \int_{36}^{27} \frac{1}{\sqrt{u}}\left(-\frac{1}{2}\,du\right) = -\frac{1}{2}\left[2\sqrt{u}\right]_{36}^{27} = -\left(\sqrt{27} - \sqrt{36}\right) = 6 - 3\sqrt{3}$$

Another method: Let $x = 6\sin\theta$, so $dx = 6\cos\theta\,d\theta$, $x = 0 \;\Rightarrow\; \theta = 0$, and $x = 3 \;\Rightarrow\; \theta = \frac{\pi}{6}$. Then

$$\int_0^3 \frac{x}{\sqrt{36-x^2}}\,dx = \int_0^{\pi/6} \frac{6\sin\theta}{\sqrt{36(1-\sin^2\theta)}}\,6\cos\theta\,d\theta = \int_0^{\pi/6} \frac{6\sin\theta}{6\cos\theta}\,6\cos\theta\,d\theta = 6\int_0^{\pi/6} \sin\theta\,d\theta$$

$$= 6\Big[-\cos\theta\Big]_0^{\pi/6} = 6\left(-\tfrac{\sqrt{3}}{2} + 1\right) = 6 - 3\sqrt{3}$$

7. Let $x = a\tan\theta$, where $a > 0$ and $-\frac{\pi}{2} < \theta < \frac{\pi}{2}$. Then $dx = a\sec^2\theta\,d\theta$, $x = 0 \;\Rightarrow\; \theta = 0$, and $x = a \;\Rightarrow\; \theta = \frac{\pi}{4}$.

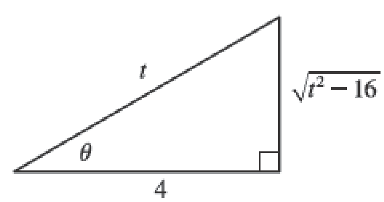
Thus,

$$\int_0^a \frac{dx}{(a^2+x^2)^{3/2}} = \int_0^{\pi/4} \frac{a\sec^2\theta\,d\theta}{[a^2(1+\tan^2\theta)]^{3/2}} = \int_0^{\pi/4} \frac{a\sec^2\theta\,d\theta}{a^3\sec^3\theta} = \frac{1}{a^2}\int_0^{\pi/4}\cos\theta\,d\theta = \frac{1}{a^2}\Big[\sin\theta\Big]_0^{\pi/4}$$

$$= \frac{1}{a^2}\left(\frac{\sqrt{2}}{2} - 0\right) = \frac{1}{\sqrt{2}\,a^2}.$$

8. Let $t = 4\sec\theta$, where $0 \le \theta < \frac{\pi}{2}$ or $\pi \le \theta < \frac{3\pi}{2}$. Then $dt = 4\sec\theta\,\tan\theta\,d\theta$ and

$\sqrt{t^2-16} = \sqrt{16\sec^2\theta - 16} = \sqrt{16\tan^2\theta} = 4\tan\theta$ for the relevant values of θ, so

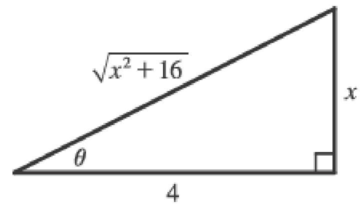


$$\int \frac{dt}{t^2\sqrt{t^2-16}} = \int \frac{4\sec\theta\,\tan\theta\,d\theta}{16\sec^2\theta\cdot 4\tan\theta} = \frac{1}{16}\int \frac{1}{\sec\theta}\,d\theta = \frac{1}{16}\int\cos\theta\,d\theta$$

$$= \frac{1}{16}\sin\theta + C = \frac{1}{16}\frac{\sqrt{t^2-16}}{t} + C = \frac{\sqrt{t^2-16}}{16t} + C$$

9. Let $x = 4\tan\theta$, where $-\frac{\pi}{2} < \theta < \frac{\pi}{2}$. Then $dx = 4\sec^2\theta\,d\theta$ and

$$\sqrt{x^2+16} = \sqrt{16\tan^2\theta + 16} = \sqrt{16(\tan^2\theta+1)}$$
$$= \sqrt{16\sec^2\theta} = 4\,|\sec\theta|$$
$$= 4\sec\theta \quad \text{for the relevant values of } \theta.$$

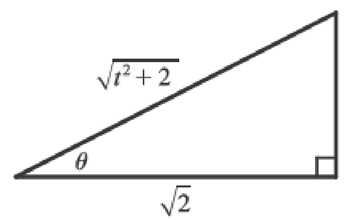


$$\int \frac{dx}{\sqrt{x^2+16}} = \int \frac{4\sec^2\theta\,d\theta}{4\sec\theta} = \int \sec\theta\,d\theta = \ln|\sec\theta+\tan\theta| + C_1 = \ln\left|\frac{\sqrt{x^2+16}}{4} + \frac{x}{4}\right| + C_1$$
$$= \ln\left|\sqrt{x^2+16}+x\right| - \ln|4| + C_1 = \ln\left(\sqrt{x^2+16}+x\right) + C, \text{ where } C = C_1 - \ln 4.$$

(Since $\sqrt{x^2+16}+x > 0$, we don't need the absolute value.)

10. Let $t = \sqrt{2}\,\tan\theta$, where $-\frac{\pi}{2} < \theta < \frac{\pi}{2}$. Then $dt = \sqrt{2}\,\sec^2\theta\,d\theta$ and

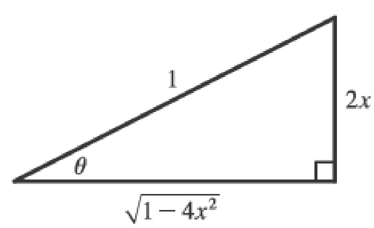
$$\sqrt{t^2+2} = \sqrt{2\tan^2\theta+2} = \sqrt{2(\tan^2\theta+1)} = \sqrt{2\sec^2\theta}$$
$$= \sqrt{2}\,|\sec\theta| = \sqrt{2}\sec\theta \text{ for the relevant values of } \theta.$$



$$\int \frac{t^5}{\sqrt{t^2+2}}\,dt = \int \frac{4\sqrt{2}\,\tan^5\theta}{\sqrt{2}\,\sec\theta}\sqrt{2}\sec^2\theta\,d\theta = 4\sqrt{2}\int \tan^5\theta\sec\theta\,d\theta$$
$$= 4\sqrt{2}\int(\sec^2\theta-1)^2\sec\theta\,\tan\theta\,d\theta = 4\sqrt{2}\int(u^2-1)^2\,du \quad [u=\sec\theta,\ du=\sec\theta\,\tan\theta\,d\theta]$$
$$= 4\sqrt{2}\int(u^4-2u^2+1)\,du = 4\sqrt{2}\left(\tfrac{1}{5}u^5 - \tfrac{2}{3}u^3 + u\right) + C$$
$$= \frac{4\sqrt{2}}{15}u(3u^4-10u^2+15) + C = \frac{4\sqrt{2}}{15}\cdot\frac{\sqrt{t^2+2}}{\sqrt{2}}\left[3\cdot\frac{(t^2+2)^2}{2^2} - 10\,\frac{t^2+2}{2} + 15\right] + C$$
$$= \tfrac{4}{15}\sqrt{t^2+2}\cdot\tfrac{1}{4}\left[3(t^4+4t^2+4) - 20(t^2+2) + 60\right] + C = \tfrac{1}{15}\sqrt{t^2+2}\,(3t^4-8t^2+32) + C$$

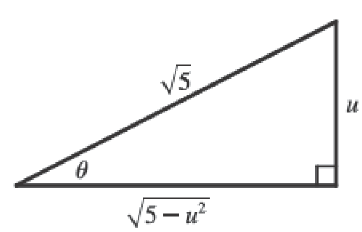
11. Let $2x = \sin\theta$, where $-\frac{\pi}{2} \le \theta \le \frac{\pi}{2}$. Then $x = \frac{1}{2}\sin\theta$, $dx = \frac{1}{2}\cos\theta\,d\theta$,

and $\sqrt{1-4x^2} = \sqrt{1-(2x)^2} = \cos\theta$.



$$\int\sqrt{1-4x^2}\,dx = \int\cos\theta\left(\tfrac{1}{2}\cos\theta\right)d\theta = \tfrac{1}{4}\int(1+\cos 2\theta)\,d\theta$$
$$= \tfrac{1}{4}\left(\theta + \tfrac{1}{2}\sin 2\theta\right) + C = \tfrac{1}{4}(\theta + \sin\theta\,\cos\theta) + C$$
$$= \tfrac{1}{4}\left[\sin^{-1}(2x) + 2x\sqrt{1-4x^2}\,\right] + C$$

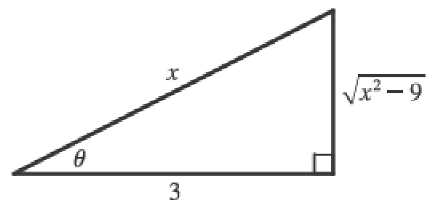
12. Let $u = \sqrt{5}\,\sin\theta$, so $du = \sqrt{5}\cos\theta\,d\theta$. Then



$$\int \frac{du}{u\sqrt{5-u^2}} = \int \frac{1}{\sqrt{5}\,\sin\theta\cdot\sqrt{5}\,\cos\theta}\sqrt{5}\,\cos\theta\,d\theta = \frac{1}{\sqrt{5}}\int\csc\theta\,d\theta$$
$$= \tfrac{1}{\sqrt{5}}\ln|\csc\theta - \cot\theta| + C \qquad \text{[by Exercise 7.2.39]}$$
$$= \frac{1}{\sqrt{5}}\ln\left|\frac{\sqrt{5}}{u} - \frac{\sqrt{5-u^2}}{u}\right| + C = \frac{1}{\sqrt{5}}\ln\left|\frac{\sqrt{5}-\sqrt{5-u^2}}{u}\right| + C$$

13. Let $x = 3\sec\theta$, where $0 \le \theta < \frac{\pi}{2}$ or $\pi \le \theta < \frac{3\pi}{2}$. Then

$dx = 3\sec\theta\,\tan\theta\,d\theta$ and $\sqrt{x^2-9} = 3\tan\theta$, so



$$\int \frac{\sqrt{x^2-9}}{x^3}\,dx = \int \frac{3\tan\theta}{27\sec^3\theta}\,3\sec\theta\,\tan\theta\,d\theta = \frac{1}{3}\int \frac{\tan^2\theta}{\sec^2\theta}d\theta$$

$$= \tfrac{1}{3}\int \sin^2\theta\,d\theta = \tfrac{1}{3}\int \tfrac{1}{2}(1-\cos 2\theta)\,d\theta = \tfrac{1}{6}\theta - \tfrac{1}{12}\sin 2\theta + C = \tfrac{1}{6}\theta - \tfrac{1}{6}\sin\theta\,\cos\theta + C$$

$$= \frac{1}{6}\sec^{-1}\left(\frac{x}{3}\right) - \frac{1}{6}\frac{\sqrt{x^2-9}}{x}\,\frac{3}{x} + C = \frac{1}{6}\sec^{-1}\left(\frac{x}{3}\right) - \frac{\sqrt{x^2-9}}{2x^2} + C$$

14. Let $x = \tan\theta$, so $dx = \sec^2\theta\,d\theta$, $x = 0 \;\Rightarrow\; \theta = 0$, and $x = 1 \;\Rightarrow\; \theta = \frac{\pi}{4}$. Then

$$\int_0^1 \frac{dx}{(x^2+1)^2} = \int_0^{\pi/4} \frac{\sec^2\theta\,d\theta}{(\tan^2\theta+1)^2} = \int_0^{\pi/4} \frac{\sec^2\theta\,d\theta}{(\sec^2\theta)^2} = \int_0^{\pi/4} \cos^2\theta\,d\theta = \int_0^{\pi/4} \frac{1}{2}\left(1+\cos 2\theta\right)\,d\theta$$

$$= \tfrac{1}{2}\left[\theta + \tfrac{1}{2}\sin 2\theta\right]_0^{\pi/4} = \tfrac{1}{2}\left[\left(\tfrac{\pi}{4}+\tfrac{1}{2}\right) - 0\right] = \tfrac{\pi}{8} + \tfrac{1}{4}$$

15. Let $x = a\sin\theta$, $dx = a\cos\theta\,d\theta$, $x = 0 \;\Rightarrow\; \theta = 0$ and $x = a \;\Rightarrow\; \theta = \frac{\pi}{2}$. Then

$$\int_0^a x^2\sqrt{a^2-x^2}\,dx = \int_0^{\pi/2} a^2\sin^2\theta\,(a\cos\theta)\,a\cos\theta\,d\theta = a^4\int_0^{\pi/2}\sin^2\theta\,\cos^2\theta\,d\theta$$

$$= a^4\int_0^{\pi/2}\left[\tfrac{1}{2}(2\sin\theta\,\cos\theta)\right]^2\,d\theta = \frac{a^4}{4}\int_0^{\pi/2}\sin^2 2\theta\,d\theta = \frac{a^4}{4}\int_0^{\pi/2}\tfrac{1}{2}(1-\cos 4\theta)\,d\theta$$

$$= \frac{a^4}{8}\left[\theta - \tfrac{1}{4}\sin 4\theta\right]_0^{\pi/2} = \frac{a^4}{8}\left[\left(\frac{\pi}{2}-0\right)-0\right] = \frac{\pi}{16}a^4$$

16. Let $x = \frac{1}{3}\sec\theta$, so $dx = \frac{1}{3}\sec\theta\,\tan\theta\,d\theta$, $x = \sqrt{2}/3 \;\Rightarrow\; \theta = \frac{\pi}{4}$, $x = \frac{2}{3} \;\Rightarrow\; \theta = \frac{\pi}{3}$. Then

$$\int_{\sqrt{2}/3}^{2/3}\frac{dx}{x^5\sqrt{9x^2-1}} = \int_{\pi/4}^{\pi/3}\frac{\frac{1}{3}\sec\theta\,\tan\theta\,d\theta}{\left(\frac{1}{3}\right)^5\sec^5\theta\,\tan\theta} = 3^4\int_{\pi/4}^{\pi/3}\cos^4\theta\,d\theta = 81\int_{\pi/4}^{\pi/3}\left[\tfrac{1}{2}(1+\cos 2\theta)\right]^2\,d\theta$$

$$= \tfrac{81}{4}\int_{\pi/4}^{\pi/3}(1+2\cos 2\theta+\cos^2 2\theta)\,d\theta = \tfrac{81}{4}\int_{\pi/4}^{\pi/3}\left[1+2\cos 2\theta+\tfrac{1}{2}(1+\cos 4\theta)\right]\,d\theta$$

$$= \tfrac{81}{4}\int_{\pi/4}^{\pi/3}\left(\tfrac{3}{2}+2\cos 2\theta+\tfrac{1}{2}\cos 4\theta\right)\,d\theta = \tfrac{81}{4}\left[\tfrac{3}{2}\theta+\sin 2\theta+\tfrac{1}{8}\sin 4\theta\right]_{\pi/4}^{\pi/3}$$

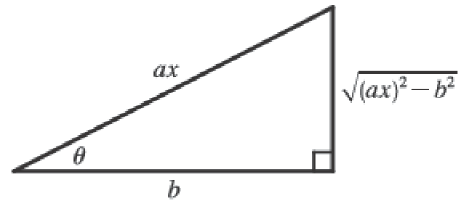
$$= \tfrac{81}{4}\left[\left(\tfrac{\pi}{2}+\tfrac{\sqrt{3}}{2}-\tfrac{\sqrt{3}}{16}\right)-\left(\tfrac{3\pi}{8}+1+0\right)\right] = \tfrac{81}{4}\left(\tfrac{\pi}{8}+\tfrac{7}{16}\sqrt{3}-1\right)$$

17. Let $u = x^2-7$, so $du = 2x\,dx$. Then $\displaystyle\int\frac{x}{\sqrt{x^2-7}}\,dx = \frac{1}{2}\int\frac{1}{\sqrt{u}}\,du = \tfrac{1}{2}\cdot 2\sqrt{u}+C = \sqrt{x^2-7}+C.$

18. Let $ax = b\sec\theta$, so $(ax)^2 = b^2\sec^2\theta \quad\Rightarrow$

$(ax)^2 - b^2 = b^2\sec^2\theta - b^2 = b^2(\sec^2\theta - 1) = b^2\tan^2\theta.$

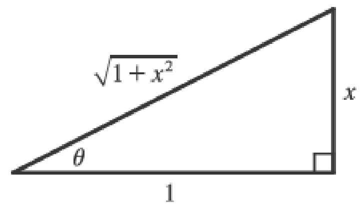
So $\sqrt{(ax)^2 - b^2} = b\tan\theta$, $dx = \dfrac{b}{a}\sec\theta\,\tan\theta\,d\theta$, and



$$\int \frac{dx}{[(ax)^2 - b^2]^{3/2}} = \int \frac{\frac{b}{a}\sec\theta\,\tan\theta}{b^3\tan^3\theta}\,d\theta = \frac{1}{ab^2}\int \frac{\sec\theta}{\tan^2\theta}\,d\theta = \frac{1}{ab^2}\int \frac{\cos\theta}{\sin^2\theta}\,d\theta = \frac{1}{ab^2}\int \csc\theta\,\cot\theta\,d\theta$$

$$= -\frac{1}{ab^2}\csc\theta + C = -\frac{1}{ab^2}\frac{ax}{\sqrt{(ax)^2 - b^2}} + C = -\frac{x}{b^2\sqrt{(ax)^2 - b^2}} + C$$

19. Let $x = \tan\theta$, where $-\frac{\pi}{2} < \theta < \frac{\pi}{2}$. Then $dx = \sec^2\theta\,d\theta$ and $\sqrt{1 + x^2} = \sec\theta$, so



$$\int \frac{\sqrt{1 + x^2}}{x}\,dx = \int \frac{\sec\theta}{\tan\theta}\,\sec^2\theta\,d\theta = \int \frac{\sec\theta}{\tan\theta}\,(1 + \tan^2\theta)\,d\theta$$

$$= \int (\csc\theta + \sec\theta\,\tan\theta)\,d\theta$$

$$= \ln|\csc\theta - \cot\theta| + \sec\theta + C \qquad \text{[by Exercise 7.2.39]}$$

$$= \ln\left|\frac{\sqrt{1 + x^2}}{x} - \frac{1}{x}\right| + \frac{\sqrt{1 + x^2}}{1} + C = \ln\left|\frac{\sqrt{1 + x^2} - 1}{x}\right| + \sqrt{1 + x^2} + C$$

20. Let $u = 1 + x^2$, so $du = 2x\,dx$. Then

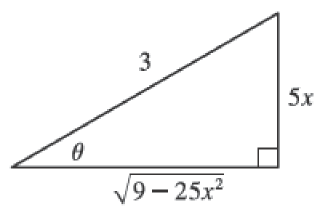
$$\int \frac{x}{\sqrt{1 + x^2}}\,dx = \int \frac{1}{\sqrt{u}}\left(\frac{1}{2}\,du\right) = \frac{1}{2}\int u^{-1/2}\,du = \frac{1}{2}\cdot 2u^{1/2} + C = \sqrt{1 + x^2} + C$$

21. Let $x = \frac{3}{5}\sin\theta$, so $dx = \frac{3}{5}\cos\theta\,d\theta$, $x = 0 \quad\Rightarrow\quad \theta = 0$, and $x = 0.6 \quad\Rightarrow\quad \theta = \frac{\pi}{2}$. Then

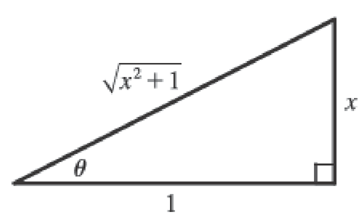
$$\int_0^{0.6} \frac{x^2}{\sqrt{9 - 25x^2}}\,dx = \int_0^{\pi/2} \frac{\left(\frac{3}{5}\right)^2\sin^2\theta}{3\cos\theta}\left(\tfrac{3}{5}\cos\theta\,d\theta\right) = \tfrac{9}{125}\int_0^{\pi/2} \sin^2\theta\,d\theta$$

$$= \tfrac{9}{125}\int_0^{\pi/2} \tfrac{1}{2}(1 - \cos 2\theta)\,d\theta = \tfrac{9}{250}\left[\theta - \tfrac{1}{2}\sin 2\theta\right]_0^{\pi/2}$$

$$= \tfrac{9}{250}\left[\left(\tfrac{\pi}{2} - 0\right) - 0\right] = \tfrac{9}{500}\pi$$



22. Let $x = \tan\theta$, where $-\frac{\pi}{2} < \theta < \frac{\pi}{2}$. Then $dx = \sec^2\theta\,d\theta$, $\sqrt{x^2 + 1} = \sec\theta$ and $x = 0 \quad\Rightarrow\quad \theta = 0$, $x = 1 \quad\Rightarrow\quad \theta = \frac{\pi}{4}$, so

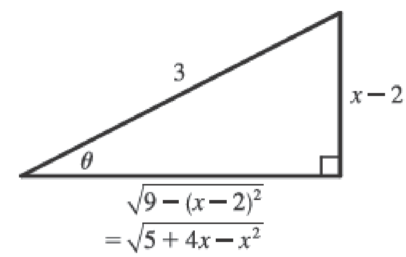


$$\int_0^1 \sqrt{x^2 + 1}\,dx = \int_0^{\pi/4} \sec\theta\,\sec^2\theta\,d\theta = \int_0^{\pi/4} \sec^3\theta\,d\theta$$

$$= \tfrac{1}{2}\Big[\sec\theta\,\tan\theta + \ln|\sec\theta + \tan\theta|\Big]_0^{\pi/4} \qquad \text{[by Example 7.2.8]}$$

$$= \tfrac{1}{2}\left[\sqrt{2}\cdot 1 + \ln\left(1 + \sqrt{2}\right) - 0 - \ln(1 + 0)\right] = \tfrac{1}{2}\left[\sqrt{2} + \ln\left(1 + \sqrt{2}\right)\right]$$

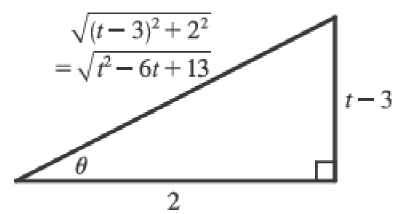
23. $5 + 4x - x^2 = -(x^2 - 4x + 4) + 9 = -(x-2)^2 + 9$. Let $x - 2 = 3\sin\theta$, $-\frac{\pi}{2} \le \theta \le \frac{\pi}{2}$, so $dx = 3\cos\theta\, d\theta$. Then



$$\begin{aligned}\int \sqrt{5+4x-x^2}\,dx &= \int \sqrt{9-(x-2)^2}\,dx = \int \sqrt{9 - 9\sin^2\theta}\,3\cos\theta\,d\theta \\ &= \int \sqrt{9\cos^2\theta}\,3\cos\theta\,d\theta = \int 9\cos^2\theta\,d\theta \\ &= \tfrac{9}{2}\int (1+\cos 2\theta)\,d\theta = \tfrac{9}{2}\left(\theta + \tfrac{1}{2}\sin 2\theta\right) + C \\ &= \tfrac{9}{2}\theta + \tfrac{9}{4}\sin 2\theta + C = \tfrac{9}{2}\theta + \tfrac{9}{4}(2\sin\theta\,\cos\theta) + C \\ &= \frac{9}{2}\sin^{-1}\left(\frac{x-2}{3}\right) + \frac{9}{2}\cdot\frac{x-2}{3}\cdot\frac{\sqrt{5+4x-x^2}}{3} + C \\ &= \frac{9}{2}\sin^{-1}\left(\frac{x-2}{3}\right) + \frac{1}{2}(x-2)\sqrt{5+4x-x^2} + C\end{aligned}$$

24. $t^2 - 6t + 13 = (t^2 - 6t + 9) + 4 = (t-3)^2 + 2^2$.

Let $t - 3 = 2\tan\theta$, so $dt = 2\sec^2\theta\,d\theta$. Then

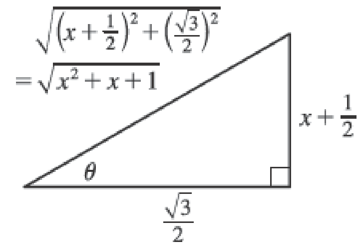


$$\begin{aligned}\int \frac{dt}{\sqrt{t^2-6t+13}} &= \int \frac{1}{\sqrt{(2\tan\theta)^2 + 2^2}}\,2\sec^2\theta\,d\theta \\ &= \int \frac{2\sec^2\theta}{2\sec\theta}\,d\theta = \int \sec\theta\,d\theta = \ln|\sec\theta + \tan\theta| + C_1 \qquad \text{[by Formula 7.2.1]} \\ &= \ln\left|\frac{\sqrt{t^2-6t+13}}{2} + \frac{t-3}{2}\right| + C_1 \\ &= \ln\left|\sqrt{t^2-6t+13} + t - 3\right| + C \qquad \text{where } C = C_1 - \ln 2\end{aligned}$$

25. $x^2+x+1=\left(x^2+x+\frac{1}{4}\right)+\frac{3}{4}=\left(x+\frac{1}{2}\right)^2+\left(\frac{\sqrt{3}}{2}\right)^2$. Let

$x+\frac{1}{2}=\frac{\sqrt{3}}{2}\tan\theta$, so $dx=\frac{\sqrt{3}}{2}\sec^2\theta\,d\theta$ and $\sqrt{x^2+x+1}=\frac{\sqrt{3}}{2}\sec\theta$.

Then



$$\int\frac{x}{\sqrt{x^2+x+1}}\,dx=\int\frac{\frac{\sqrt{3}}{2}\tan\theta-\frac{1}{2}}{\frac{\sqrt{3}}{2}\sec\theta}\,\frac{\sqrt{3}}{2}\,\sec^2\theta\,d\theta$$

$$=\int\left(\frac{\sqrt{3}}{2}\tan\theta-\frac{1}{2}\right)\sec\theta\,d\theta=\int\frac{\sqrt{3}}{2}\tan\theta\,\sec\theta\,d\theta-\int\frac{1}{2}\sec\theta\,d\theta$$

$$=\frac{\sqrt{3}}{2}\sec\theta-\frac{1}{2}\ln|\sec\theta+\tan\theta|+C_1$$

$$=\sqrt{x^2+x+1}-\frac{1}{2}\ln\left|\frac{2}{\sqrt{3}}\sqrt{x^2+x+1}+\frac{2}{\sqrt{3}}\left(x+\frac{1}{2}\right)\right|+C_1$$

$$=\sqrt{x^2+x+1}-\frac{1}{2}\ln\left|\frac{2}{\sqrt{3}}\left[\sqrt{x^2+x+1}+\left(x+\frac{1}{2}\right)\right]\right|+C_1$$

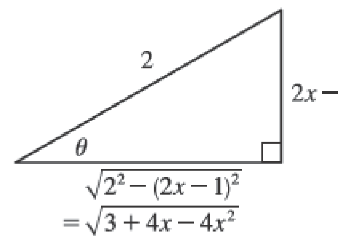
$$=\sqrt{x^2+x+1}-\frac{1}{2}\ln\frac{2}{\sqrt{3}}-\frac{1}{2}\ln\left(\sqrt{x^2+x+1}+x+\frac{1}{2}\right)+C_1$$

$$=\sqrt{x^2+x+1}-\frac{1}{2}\ln\left(\sqrt{x^2+x+1}+x+\frac{1}{2}\right)+C,\quad\text{where } C=C_1-\frac{1}{2}\ln\frac{2}{\sqrt{3}}$$

26. $3+4x-4x^2=-(4x^2-4x+1)+4=2^2-(2x-1)^2$.

Let $2x-1=2\sin\theta$, so $2\,dx=2\cos\theta\,d\theta$ and $\sqrt{3+4x-4x^2}=2\cos\theta$.

Then



$$\int\frac{x^2}{(3+4x-4x^2)^{3/2}}\,dx=\int\frac{\left[\frac{1}{2}(1+2\sin\theta)\right]^2}{(2\cos\theta)^3}\cos\theta\,d\theta$$

$$=\frac{1}{32}\int\frac{1+4\sin\theta+4\sin^2\theta}{\cos^2\theta}\,d\theta=\frac{1}{32}\int(\sec^2\theta+4\tan\theta\,\sec\theta+4\tan^2\theta)\,d\theta$$

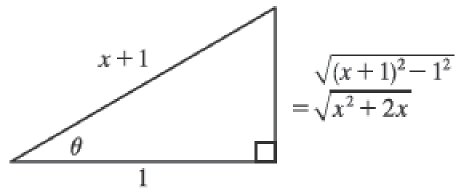
$$=\frac{1}{32}\int[\sec^2\theta+4\tan\theta\,\sec\theta+4(\sec^2\theta-1)]\,d\theta$$

$$=\frac{1}{32}\int(5\sec^2\theta+4\tan\theta\,\sec\theta-4)\,d\theta=\frac{1}{32}(5\tan\theta+4\sec\theta-4\theta)+C$$

$$=\frac{1}{32}\left[5\cdot\frac{2x-1}{\sqrt{3+4x-4x^2}}+4\cdot\frac{2}{\sqrt{3+4x-4x^2}}-4\cdot\sin^{-1}\left(\frac{2x-1}{2}\right)\right]+C$$

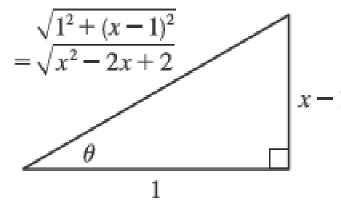
$$=\frac{10x+3}{32\sqrt{3+4x-4x^2}}-\frac{1}{8}\sin^{-1}\left(\frac{2x-1}{2}\right)+C$$

27. $x^2+2x=(x^2+2x+1)-1=(x+1)^2-1$. Let $x+1=1\sec\theta$, so $dx=\sec\theta\,\tan\theta\,d\theta$ and $\sqrt{x^2+2x}=\tan\theta$. Then



$$\int\sqrt{x^2+2x}\,dx=\int\tan\theta\,(\sec\theta\,\tan\theta\,d\theta)=\int\tan^2\theta\,\sec\theta\,d\theta$$
$$=\int(\sec^2\theta-1)\,\sec\theta\,d\theta=\int\sec^3\theta\,d\theta-\int\sec\theta\,d\theta$$
$$=\tfrac{1}{2}\sec\theta\,\tan\theta+\tfrac{1}{2}\ln|\sec\theta+\tan\theta|-\ln|\sec\theta+\tan\theta|+C$$
$$=\tfrac{1}{2}\sec\theta\,\tan\theta-\tfrac{1}{2}\ln|\sec\theta+\tan\theta|+C=\tfrac{1}{2}(x+1)\sqrt{x^2+2x}-\tfrac{1}{2}\ln\left|x+1+\sqrt{x^2+2x}\right|+C$$

28. $x^2-2x+2=(x^2-2x+1)+1=(x-1)^2+1$. Let $x-1=1\tan\theta$, so $dx=\sec^2\theta\,d\theta$ and $\sqrt{x^2-2x+2}=\sec\theta$. Then



$$\int\frac{x^2+1}{(x^2-2x+2)^2}\,dx=\int\frac{(\tan\theta+1)^2+1}{\sec^4\theta}\,\sec^2\theta\,d\theta$$
$$=\int\frac{\tan^2\theta+2\tan\theta+2}{\sec^2\theta}\,d\theta$$
$$=\int(\sin^2\theta+2\sin\theta\,\cos\theta+2\cos^2\theta)\,d\theta=\int(1+2\sin\theta\,\cos\theta+\cos^2\theta)\,d\theta$$
$$=\int\left[1+2\sin\theta\,\cos\theta+\tfrac{1}{2}(1+\cos 2\theta)\right]\,d\theta=\int\left(\tfrac{3}{2}+2\sin\theta\,\cos\theta+\tfrac{1}{2}\cos 2\theta\right)\,d\theta$$
$$=\tfrac{3}{2}\theta+\sin^2\theta+\tfrac{1}{4}\sin 2\theta+C=\tfrac{3}{2}\theta+\sin^2\theta+\tfrac{1}{2}\sin\theta\,\cos\theta+C$$
$$=\frac{3}{2}\tan^{-1}\left(\frac{x-1}{1}\right)+\frac{(x-1)^2}{x^2-2x+2}+\frac{1}{2}\,\frac{x-1}{\sqrt{x^2-2x+2}}\,\frac{1}{\sqrt{x^2-2x+2}}+C$$
$$=\frac{3}{2}\tan^{-1}(x-1)+\frac{2(x^2-2x+1)+x-1}{2(x^2-2x+2)}+C=\frac{3}{2}\tan^{-1}(x-1)+\frac{2x^2-3x+1}{2(x^2-2x+2)}+C$$

We can write the answer as

$$\frac{3}{2}\tan^{-1}(x-1)+\frac{(2x^2-4x+4)+x-3}{2(x^2-2x+2)}+C=\frac{3}{2}\tan^{-1}(x-1)+1+\frac{x-3}{2(x^2-2x+2)}+C$$
$$=\frac{3}{2}\tan^{-1}(x-1)+\frac{x-3}{2(x^2-2x+2)}+C_1,\ \text{where } C_1=1+C$$

29. Let $u=x^2$, $du=2x\,dx$. Then

$$\int x\sqrt{1-x^4}\,dx=\int\sqrt{1-u^2}\left(\tfrac{1}{2}\,du\right)=\tfrac{1}{2}\int\cos\theta\cdot\cos\theta\,d\theta\qquad\left[\begin{array}{c}\text{where } u=\sin\theta,\ du=\cos\theta\,d\theta,\\ \text{and } \sqrt{1-u^2}=\cos\theta\end{array}\right]$$
$$=\tfrac{1}{2}\int\tfrac{1}{2}(1+\cos 2\theta)\,d\theta=\tfrac{1}{4}\theta+\tfrac{1}{8}\sin 2\theta+C=\tfrac{1}{4}\theta+\tfrac{1}{4}\sin\theta\,\cos\theta+C$$
$$=\tfrac{1}{4}\sin^{-1}u+\tfrac{1}{4}u\sqrt{1-u^2}+C=\tfrac{1}{4}\sin^{-1}(x^2)+\tfrac{1}{4}x^2\sqrt{1-x^4}+C$$

30. Let $u = \sin t$, $du = \cos t\, dt$. Then

$$\int_0^{\pi/2} \frac{\cos t}{\sqrt{1+\sin^2 t}}\, dt = \int_0^1 \frac{1}{\sqrt{1+u^2}}\, du = \int_0^{\pi/4} \frac{1}{\sec\theta}\ \sec^2\theta\, d\theta \qquad \left[\begin{array}{c}\text{where } u = \tan\theta\text{ , } du = \sec^2\theta\, d\theta, \\ \text{and } \sqrt{1+u^2} = \sec\theta\end{array}\right]$$

$$= \int_0^{\pi/4} \sec\theta\, d\theta = \Big[\ln|\sec\theta + \tan\theta|\Big]_0^{\pi/4} \qquad \text{[by (1) in Section 7.2]}$$

$$= \ln\big(\sqrt{2}+1\big) - \ln(1+0) = \ln\big(\sqrt{2}+1\big)$$