1. $\int \sin^2 x \, \cos^3 x \, dx = \int \sin^2 x \, \cos^2 x \, \cos x \, dx = \int \sin^2 x \, (1 - \sin^2 x) \, \cos x \, dx$

$$\overset{s}{=} \int u^2 (1 - u^2) \, du = \int (u^2 - u^4) \, du = \tfrac{1}{3} u^3 - \tfrac{1}{5} u^5 + C = \tfrac{1}{3} \sin^3 x - \tfrac{1}{5} \sin^5 x + C$$

2. $\int \sin^3 \theta \, \cos^4 \theta \, d\theta = \int \sin^2 \theta \, \cos^4 \theta \, \sin \theta \, d\theta = \int (1 - \cos^2 \theta) \, \cos^4 \theta \, \sin \theta \, d\theta$

$$\overset{c}{=} \int (1 - u^2) u^4 (-du) = \int (u^6 - u^4) \, du = \tfrac{1}{7} u^7 - \tfrac{1}{5} u^5 + C = \tfrac{1}{7} \cos^7 \theta - \tfrac{1}{5} \cos^5 \theta + C$$

3. $\int_0^{\pi/2} \sin^7 \theta \, \cos^5 \theta \, d\theta = \int_0^{\pi/2} \sin^7 \theta \, \cos^4 \theta \, \cos \theta \, d\theta = \int_0^{\pi/2} \sin^7 \theta \, (1 - \sin^2 \theta)^2 \, \cos \theta \, d\theta$

$$\overset{s}{=} \int_0^1 u^7 (1 - u^2)^2 \, du = \int_0^1 u^7 (1 - 2u^2 + u^4) \, du = \int_0^1 (u^7 - 2u^9 + u^{11}) \, du$$

$$= \left[\frac{1}{8} u^8 - \frac{1}{5} u^{10} + \frac{1}{12} u^{12} \right]_0^1 = \left(\frac{1}{8} - \frac{1}{5} + \frac{1}{12} \right) - 0 = \frac{15 - 24 + 10}{120} = \frac{1}{120}$$

4. $\int_0^{\pi/2} \sin^5 x \, dx = \int_0^{\pi/2} \sin^4 x \, \sin x \, dx = \int_0^{\pi/2} (1 - \cos^2 x)^2 \, \sin x \, dx \overset{c}{=} \int_1^0 (1 - u^2)^2 (-du)$

$$= \int_0^1 (1 - 2u^2 + u^4) \, du = \left[u - \frac{2}{3} u^3 + \frac{1}{5} u^5 \right]_0^1 = \left(1 - \frac{2}{3} + \frac{1}{5} \right) - 0 = \frac{15 - 10 + 3}{15} = \frac{8}{15}$$

5. Let $y = \pi x$, so $dy = \pi \, dx$ and

$$\int \sin^2 (\pi x) \, \cos^5 (\pi x) \, dx = \tfrac{1}{\pi} \int \sin^2 y \, \cos^5 y \, dy = \tfrac{1}{\pi} \int \sin^2 y \, \cos^4 y \, \cos y \, dy$$

$$= \tfrac{1}{\pi} \int \sin^2 y \, (1 - \sin^2 y)^2 \, \cos y \, dy \overset{s}{=} \tfrac{1}{\pi} \int u^2 (1 - u^2)^2 \, du = \tfrac{1}{\pi} \int (u^2 - 2u^4 + u^6) \, du$$

$$= \tfrac{1}{\pi} \left(\tfrac{1}{3} u^3 - \tfrac{2}{5} u^5 + \tfrac{1}{7} u^7 \right) + C = \tfrac{1}{3\pi} \sin^3 y - \tfrac{2}{5\pi} \sin^5 y + \tfrac{1}{7\pi} \sin^7 y + C$$

$$= \tfrac{1}{3\pi} \sin^3 (\pi x) - \tfrac{2}{5\pi} \sin^5 (\pi x) + \tfrac{1}{7\pi} \sin^7 (\pi x) + C$$

6. Let $y = \sqrt{x}$, so that $dy = \dfrac{1}{2\sqrt{x}} \, dx$ and $dx = 2y \, dy$. Then

$$\int \frac{\sin^3 (\sqrt{x})}{\sqrt{x}} \, dx = \int \frac{\sin^3 y}{y} \, (2y \, dy) = 2 \int \sin^3 y \, dy = 2 \int \sin^2 y \, \sin y \, dy = 2 \int (1 - \cos^2 y) \, \sin y \, dy$$

$$\overset{c}{=} 2 \int (1 - u^2)(-du) = 2 \int (u^2 - 1) \, du = 2 \left(\tfrac{1}{3} u^3 - u \right) + C = \tfrac{2}{3} \cos^3 y - 2 \cos y + C$$

$$= \tfrac{2}{3} \cos^3 (\sqrt{x}) - 2 \cos \sqrt{x} + C$$

7. $\int_0^{\pi/2} \cos^2 \theta \, d\theta = \int_0^{\pi/2} \tfrac{1}{2} (1 + \cos 2\theta) \, d\theta$ [half-angle identity]

$$= \tfrac{1}{2} \left[\theta + \tfrac{1}{2} \sin 2\theta \right]_0^{\pi/2} = \tfrac{1}{2} \left[\left(\tfrac{\pi}{2} + 0 \right) - (0 + 0) \right] = \tfrac{\pi}{4}$$

8. $\int_0^{2\pi} \sin^2 \left(\tfrac{1}{3} \theta \right) d\theta = \int_0^{2\pi} \tfrac{1}{2} \left[1 - \cos \left(2 \cdot \tfrac{1}{3} \theta \right) \right] d\theta$ [half-angle identity]

$$= \frac{1}{2} \left[\theta - \frac{3}{2} \sin \left(\frac{2}{3} \theta \right) \right]_0^{2\pi} = \frac{1}{2} \left[\left(2\pi - \frac{3}{2} \left(-\frac{\sqrt{3}}{2} \right) \right) - 0 \right] = \pi + \frac{3}{8} \sqrt{3}$$

9. $\int_0^\pi \cos^4(2t)\,dt = \int_0^\pi [\cos^2(2t)]^2\,dt = \int_0^\pi \left[\frac{1}{2}(1+\cos(2\cdot 2t))\right]^2 dt$ [half-angle identity]

$$= \frac{1}{4}\int_0^\pi [1 + 2\cos 4t + \cos^2(4t)]\,dt = \frac{1}{4}\int_0^\pi [1 + 2\cos 4t + \frac{1}{2}(1+\cos 8t)]\,dt$$

$$= \frac{1}{4}\int_0^\pi \left(\frac{3}{2} + 2\cos 4t + \frac{1}{2}\cos 8t\right)\,dt = \frac{1}{4}\left[\frac{3}{2}t + \frac{1}{2}\sin 4t + \frac{1}{16}\sin 8t\right]_0^\pi = \frac{1}{4}\left[\left(\frac{3}{2}\pi + 0 + 0\right) - 0\right] = \frac{3}{8}\pi$$

10. $\int_0^\pi \sin^2 t\,\cos^4 t\,dt = \frac{1}{4}\int_0^\pi (4\sin^2 t\,\cos^2 t)\cos^2 t\,dt = \frac{1}{4}\int_0^\pi (2\sin t\,\cos t)^2\,\frac{1}{2}(1+\cos 2t)\,dt$

$$= \frac{1}{8}\int_0^\pi (\sin 2t)^2(1+\cos 2t)\,dt = \frac{1}{8}\int_0^\pi (\sin^2 2t + \sin^2 2t\,\cos 2t)\,dt$$

$$= \frac{1}{8}\int_0^\pi \sin^2 2t\,dt + \frac{1}{8}\int_0^\pi \sin^2 2t\,\cos 2t\,dt = \frac{1}{8}\int_0^\pi \frac{1}{2}(1-\cos 4t)\,dt + \frac{1}{8}\left[\frac{1}{3}\cdot\frac{1}{2}\sin^3 2t\right]_0^\pi$$

$$= \frac{1}{16}\left[t - \frac{1}{4}\sin 4t\right]_0^\pi + \frac{1}{8}(0-0) = \frac{1}{16}\left[(\pi - 0) - 0\right] = \frac{\pi}{16}$$

11. $\int_0^{\pi/2} \sin^2 x\,\cos^2 x\,dx = \int_0^{\pi/2} \frac{1}{4}(4\sin^2 x\,\cos^2 x)\,dx = \int_0^{\pi/2}\frac{1}{4}(2\sin x\,\cos x)^2 dx = \frac{1}{4}\int_0^{\pi/2}\sin^2 2x\,dx$

$$= \frac{1}{4}\int_0^{\pi/2}\frac{1}{2}(1-\cos 4x)\,dx = \frac{1}{8}\int_0^{\pi/2}(1-\cos 4x)\,dx = \frac{1}{8}\left[x - \frac{1}{4}\sin 4x\right]_0^{\pi/2} = \frac{1}{8}\left(\frac{\pi}{2}\right) = \frac{\pi}{16}$$

12. $\int_0^{\pi/2}(2-\sin\theta)^2\,d\theta = \int_0^{\pi/2}(4 - 4\sin\theta + \sin^2\theta)\,d\theta = \int_0^{\pi/2}\left[4 - 4\sin\theta + \frac{1}{2}(1-\cos 2\theta)\right]d\theta$

$$= \int_0^{\pi/2}\left(\frac{9}{2} - 4\sin\theta - \frac{1}{2}\cos 2\theta\right)d\theta = \left[\frac{9}{2}\theta + 4\cos\theta - \frac{1}{4}\sin 2\theta\right]_0^{\pi/2}$$

$$= \left(\frac{9\pi}{4} + 0 - 0\right) - (0 + 4 - 0) = \frac{9}{4}\pi - 4$$

13. $\int t\sin^2 t\,dt = \int t\left[\frac{1}{2}(1-\cos 2t)\right]\,dt = \frac{1}{2}\int(t - t\cos 2t)\,dt = \frac{1}{2}\int t\,dt - \frac{1}{2}\int t\cos 2t\,dt$

$$= \frac{1}{2}\left(\frac{1}{2}t^2\right) - \frac{1}{2}\left(\frac{1}{2}t\sin 2t - \int\frac{1}{2}\sin 2t\,dt\right) \qquad \begin{bmatrix} u = t, & dv = \cos 2t\,dt \\ du = dt, & v = \frac{1}{2}\sin 2t \end{bmatrix}$$

$$= \frac{1}{4}t^2 - \frac{1}{4}t\sin 2t + \frac{1}{2}\left(-\frac{1}{4}\cos 2t\right) + C = \frac{1}{4}t^2 - \frac{1}{4}t\sin 2t - \frac{1}{8}\cos 2t + C$$

14. Let $u = \sin\theta$. Then $du = \cos\theta\,d\theta$ and

$$\int\cos\theta\,\cos^5(\sin\theta)\,d\theta = \int\cos^5 u\,du = \int(\cos^2 u)^2\cos u\,du = \int(1-\sin^2 u)^2\cos u\,du$$

$$= \int(1 - 2\sin^2 u + \sin^4 u)\cos u\,du = I$$

Now let $x = \sin u$. Then $dx = \cos u\,du$ and

$$I = \int(1 - 2x^2 + x^4)\,dx = x - \frac{2}{3}x^3 + \frac{1}{5}x^5 + C = \sin u - \frac{2}{3}\sin^3 u + \frac{1}{5}\sin^5 u + C$$

$$= \sin(\sin\theta) - \frac{2}{3}\sin^3(\sin\theta) + \frac{1}{5}\sin^5(\sin\theta) + C$$

15. $\displaystyle\int\frac{\cos^5\alpha}{\sqrt{\sin\alpha}}\,d\alpha = \int\frac{\cos^4\alpha}{\sqrt{\sin\alpha}}\cos\alpha\,d\alpha = \int\frac{\left(1-\sin^2\alpha\right)^2}{\sqrt{\sin\alpha}}\cos\alpha\,d\alpha \overset{s}{=} \int\frac{(1-u^2)^2}{\sqrt{u}}\,du$

$$= \int\frac{1 - 2u^2 + u^4}{u^{1/2}}\,du = \int(u^{-1/2} - 2u^{3/2} + u^{7/2})\,du = 2u^{1/2} - \frac{4}{5}u^{5/2} + \frac{2}{9}u^{9/2} + C$$

$$= \frac{2}{45}u^{1/2}(45 - 18u^2 + 5u^4) + C = \frac{2}{45}\sqrt{\sin\alpha}\,(45 - 18\sin^2\alpha + 5\sin^4\alpha) + C$$

16. $I = \int x \sin^3 x\, dx$. First, evaluate

$\int \sin^3 x\, dx = \int (1 - \cos^2 x)\, \sin x\, dx \overset{c}{=} \int (1 - u^2)(-du) = \int (u^2 - 1)\, du = \frac{1}{3}u^3 - u + C_1 = \frac{1}{3}\cos^3 x - \cos x + C_1.$

Now for I, let $u = x$, $dv = \sin^3 x \quad \Rightarrow \quad du = dx$, $v = \frac{1}{3}\cos^3 x - \cos x$, so

$$\begin{aligned} I &= \tfrac{1}{3}x\cos^3 x - x\cos x - \int \left(\tfrac{1}{3}\cos^3 x - \cos x\right) dx = \tfrac{1}{3}x\cos^3 x - x\cos x - \tfrac{1}{3}\int \cos^3 x\, dx + \sin x \\ &= \tfrac{1}{3}x\cos^3 x - x\cos x - \tfrac{1}{3}(\sin x - \tfrac{1}{3}\sin^3 x) + \sin x + C \qquad \text{[by Example 1]} \\ &= \tfrac{1}{3}x\cos^3 x - x\cos x + \tfrac{2}{3}\sin x + \tfrac{1}{9}\sin^3 x + C \end{aligned}$$

17. $$\begin{aligned} \int \cos^2 x\, \tan^3 x\, dx &= \int \frac{\sin^3 x}{\cos x}\, dx \overset{c}{=} \int \frac{(1-u^2)(-du)}{u} = \int \left[\frac{-1}{u} + u\right] du \\ &= -\ln|u| + \tfrac{1}{2}u^2 + C = \tfrac{1}{2}\cos^2 x - \ln|\cos x| + C \end{aligned}$$

18. $$\begin{aligned} \int \cot^5\theta \sin^4\theta\, d\theta &= \int \frac{\cos^5\theta}{\sin^5\theta}\, \sin^4\theta\, d\theta = \int \frac{\cos^5\theta}{\sin\theta}\, d\theta = \int \frac{\cos^4\theta}{\sin\theta}\, \cos\theta\, d\theta = \int \frac{\left(1-\sin^2\theta\right)^2}{\sin\theta}\, \cos\theta\, d\theta \\ &\overset{s}{=} \int \frac{\left(1-u^2\right)^2}{u}\, du = \int \frac{1 - 2u^2 + u^4}{u}\, du = \int \left(\frac{1}{u} - 2u + u^3\right) du \\ &= \ln|u| - u^2 + \tfrac{1}{4}u^4 + C = \ln|\sin\theta| - \sin^2\theta + \tfrac{1}{4}\sin^4\theta + C \end{aligned}$$

19. $$\begin{aligned} \int \frac{\cos x + \sin 2x}{\sin x}\, dx &= \int \frac{\cos x + 2\sin x\, \cos x}{\sin x}\, dx = \int \frac{\cos x}{\sin x}\, dx + \int 2\cos x\, dx \overset{s}{=} \int \frac{1}{u}\, du + 2\sin x \\ &= \ln|u| + 2\sin x + C = \ln|\sin x| + 2\sin x + C \end{aligned}$$

Or: Use the formula $\int \cot x\, dx = \ln|\sin x| + C$.

20. $\int \cos^2 x\, \sin 2x\, dx = 2\int \cos^3 x\, \sin x\, dx \overset{c}{=} -2\int u^3\, du = -\frac{1}{2}u^4 + C = -\frac{1}{2}\cos^4 x + C$

21. $$\begin{aligned} \int \tan x\, \sec^3 x\, dx &= \int \tan x\, \sec x\, \sec^2 x\, dx = \int u^2\, du \qquad [u = \sec x,\, du = \sec x\, \tan x\, dx] \\ &= \tfrac{1}{3}u^3 + C = \tfrac{1}{3}\sec^3 x + C \end{aligned}$$

22. $$\begin{aligned} \int \tan^2\theta\, \sec^4\theta\, d\theta &= \int \tan^2\theta\, \sec^2\theta\, \sec^2\theta\, d\theta = \int \tan^2\theta\, (\tan^2\theta + 1)\, \sec^2\theta\, d\theta \\ &= \int u^2(u^2+1)\, du \qquad [u = \tan\theta,\, du = \sec^2\theta\, d\theta] \\ &= \int (u^4 + u^2)\, du = \tfrac{1}{5}u^5 + \tfrac{1}{3}u^3 + C = \tfrac{1}{5}\tan^5\theta + \tfrac{1}{3}\tan^3\theta + C \end{aligned}$$

23. $\int \tan^2 x\, dx = \int (\sec^2 x - 1)\, dx = \tan x - x + C$

24. $$\begin{aligned} \int (\tan^2 x + \tan^4 x)\, dx &= \int \tan^2 x\, (1 + \tan^2 x)\, dx = \int \tan^2 x\, \sec^2 x\, dx = \int u^2\, du \qquad [u = \tan x,\, du = \sec^2 x\, dx] \\ &= \tfrac{1}{3}u^3 + C = \tfrac{1}{3}\tan^3 x + C \end{aligned}$$

25. Let $u = \tan x$. Then $du = \sec^2 x\,dx$, so

$$\begin{aligned}\int \tan^4 x \sec^6 x\,dx &= \int \tan^4 x \sec^4 x\,(\sec^2 x\,dx) = \int \tan^4 x (1+\tan^2 x)^2\,(\sec^2 x\,dx)\\ &= \int u^4(1+u^2)^2\,du = \int (u^8 + 2u^6 + u^4)\,du\\ &= \tfrac{1}{9}u^9 + \tfrac{2}{7}u^7 + \tfrac{1}{5}u^5 + C = \tfrac{1}{9}\tan^9 x + \tfrac{2}{7}\tan^7 x + \tfrac{1}{5}\tan^5 x + C\end{aligned}$$

26. $\int_0^{\pi/4} \sec^4\theta\,\tan^4\theta\,d\theta = \int_0^{\pi/4} (\tan^2\theta + 1)\,\tan^4\theta\,\sec^2\theta\,d\theta = \int_0^1 (u^2+1)u^4\,du \qquad [u = \tan\theta,\ du = \sec^2\theta\,d\theta]$

$$= \int_0^1 (u^6 + u^4)\,du = \left[\tfrac{1}{7}u^7 + \tfrac{1}{5}u^5\right]_0^1 = \tfrac{1}{7} + \tfrac{1}{5} = \tfrac{12}{35}$$

27. $\int_0^{\pi/3} \tan^5 x\,\sec^4 x\,dx = \int_0^{\pi/3} \tan^5 x\,(\tan^2 x + 1)\sec^2 x\,dx = \int_0^{\sqrt{3}} u^5(u^2+1)\,du \qquad [u = \tan x,\ du = \sec^2 x\,dx]$

$$= \int_0^{\sqrt{3}} (u^7 + u^5)\,du = \left[\tfrac{1}{8}u^8 + \tfrac{1}{6}u^6\right]_0^{\sqrt{3}} = \tfrac{81}{8} + \tfrac{27}{6} = \tfrac{81}{8} + \tfrac{9}{2} = \tfrac{81}{8} + \tfrac{36}{8} = \tfrac{117}{8}$$

Alternate solution:

$$\begin{aligned}\int_0^{\pi/3} \tan^5 x\,\sec^4 x\,dx &= \int_0^{\pi/3} \tan^4 x\,\sec^3 x\,\sec x\,\tan x\,dx = \int_0^{\pi/3} (\sec^2 x - 1)^2\,\sec^3 x\,\sec x\,\tan x\,dx\\ &= \int_1^2 (u^2-1)^2 u^3\,du \quad [u = \sec x,\ du = \sec x\,\tan x\,dx] \quad = \int_1^2 (u^4 - 2u^2 + 1)u^3\,du\\ &= \int_1^2 (u^7 - 2u^5 + u^3)\,du = \left[\tfrac{1}{8}u^8 - \tfrac{1}{3}u^6 + \tfrac{1}{4}u^4\right]_1^2 = \left(32 - \tfrac{64}{3} + 4\right) - \left(\tfrac{1}{8} - \tfrac{1}{3} + \tfrac{1}{4}\right) = \tfrac{117}{8}\end{aligned}$$

28. Let $u = \sec x$, so $du = \sec x \tan x\,dx$. Thus,

$$\begin{aligned}\int \tan^5 x \sec^3 x\,dx &= \int \tan^4 x \sec^2 x\,(\sec x \tan x)\,dx = \int (\sec^2 x - 1)^2 \sec^2 x\,(\sec x \tan x\,dx)\\ &= \int (u^2-1)^2 u^2\,du = \int (u^6 - 2u^4 + u^2)\,du\\ &= \tfrac{1}{7}u^7 - \tfrac{2}{5}u^5 + \tfrac{1}{3}u^3 + C = \tfrac{1}{7}\sec^7 x - \tfrac{2}{5}\sec^5 x + \tfrac{1}{3}\sec^3 x + C\end{aligned}$$

29. $\int \tan^3 x\,\sec x\,dx = \int \tan^2 x\,\sec x\,\tan x\,dx = \int (\sec^2 x - 1)\,\sec x\,\tan x\,dx$

$$= \int (u^2 - 1)\,du \quad [u = \sec x,\ du = \sec x\,\tan x\,dx] \quad = \tfrac{1}{3}u^3 - u + C = \tfrac{1}{3}\sec^3 x - \sec x + C$$

30. $\int_0^{\pi/4} \tan^4 t\,dt = \int_0^{\pi/4} \tan^2 t\,(\sec^2 t - 1)\,dt = \int_0^{\pi/4} \tan^2 t\,\sec^2 t\,dt - \int_0^{\pi/4} \tan^2 t\,dt$

$$\begin{aligned}&= \int_0^1 u^2\,du \ [u = \tan t] \ - \int_0^{\pi/4} (\sec^2 t - 1)\,dt = \left[\tfrac{1}{3}u^3\right]_0^1 - \left[\tan t - t\right]_0^{\pi/4}\\ &= \tfrac{1}{3} - \left[\left(1 - \tfrac{\pi}{4}\right) - 0\right] = \tfrac{\pi}{4} - \tfrac{2}{3}\end{aligned}$$

31. $\int \tan^5 x\,dx = \int (\sec^2 x - 1)^2\,\tan x\,dx = \int \sec^4 x\,\tan x\,dx - 2\int \sec^2 x\,\tan x\,dx + \int \tan x\,dx$

$$\begin{aligned}&= \int \sec^3 x\,\sec x\,\tan x\,dx - 2\int \tan x\,\sec^2 x\,dx + \int \tan x\,dx\\ &= \tfrac{1}{4}\sec^4 x - \tan^2 x + \ln|\sec x| + C \quad \left[\text{or } \tfrac{1}{4}\sec^4 x - \sec^2 x + \ln|\sec x| + C\right]\end{aligned}$$

32. $\int \tan^2 x \sec x\, dx = \int (\sec^2 x - 1) \sec x\, dx = \int \sec^3 x\, dx - \int \sec x\, dx$

$= \frac{1}{2}(\sec x \tan x + \ln|\sec x + \tan x|) - \ln|\sec x + \tan x| + C$ [by Example 8 and (1)]

$= \frac{1}{2}(\sec x \tan x - \ln|\sec x + \tan x|) + C$

33. Let $u = x, dv = \sec x \tan x\, dx \Rightarrow du = dx, v = \sec x$. Then

$\int x \sec x \tan x\, dx = x \sec x - \int \sec x\, dx = x \sec x - \ln|\sec x + \tan x| + C$.

34. $\displaystyle\int \frac{\sin\phi}{\cos^3\phi}\, d\phi = \int \frac{\sin\phi}{\cos\phi} \cdot \frac{1}{\cos^2\phi}\, d\phi = \int \tan\phi \sec^2\phi\, d\phi = \int u\, du$ $[u = \tan\phi, du = \sec^2\phi\, d\phi]$

$= \frac{1}{2}u^2 + C = \frac{1}{2}\tan^2\phi + C$

Alternate solution: Let $u = \cos\phi$ to get $\frac{1}{2}\sec^2\phi + C$.

35. $\int_{\pi/6}^{\pi/2} \cot^2 x\, dx = \int_{\pi/6}^{\pi/2} (\csc^2 x - 1)\, dx = [-\cot x - x]_{\pi/6}^{\pi/2} = \left(0 - \frac{\pi}{2}\right) - \left(-\sqrt{3} - \frac{\pi}{6}\right) = \sqrt{3} - \frac{\pi}{3}$

36. $\displaystyle\int_{\pi/4}^{\pi/2} \cot^3 x\, dx = \int_{\pi/4}^{\pi/2} \cot x (\csc^2 x - 1)\, dx = \int_{\pi/4}^{\pi/2} \cot x \csc^2 x\, dx - \int_{\pi/4}^{\pi/2} \frac{\cos x}{\sin x}\, dx$

$= \left[-\frac{1}{2}\cot^2 x - \ln|\sin x|\right]_{\pi/4}^{\pi/2} = (0 - \ln 1) - \left[-\frac{1}{2} - \ln\frac{1}{\sqrt{2}}\right] = \frac{1}{2} + \ln\frac{1}{\sqrt{2}} = \frac{1}{2}(1 - \ln 2)$

37.
$\int_{\pi/4}^{\pi/2} \cot^5\phi \csc^3\phi\, d\phi = \int_{\pi/4}^{\pi/2} \cot^4\phi \csc^2\phi \csc\phi \cot\phi\, d\phi = \int_{\pi/4}^{\pi/2} (\csc^2\phi - 1)^2 \csc^2\phi \csc\phi \cot\phi\, d\phi$

$\displaystyle= \int_{\sqrt{2}}^{1} (u^2 - 1)^2 u^2 (-du)$ $[u = \csc\phi, du = -\csc\phi \cot\phi\, d\phi]$

$\displaystyle= \int_{1}^{\sqrt{2}} (u^6 - 2u^4 + u^2)\, du = \left[\frac{1}{7}u^7 - \frac{2}{5}u^5 + \frac{1}{3}u^3\right]_1^{\sqrt{2}} = \left(\frac{8}{7}\sqrt{2} - \frac{8}{5}\sqrt{2} + \frac{2}{3}\sqrt{2}\right) - \left(\frac{1}{7} - \frac{2}{5} + \frac{1}{3}\right)$

$\displaystyle= \frac{120 - 168 + 70}{105}\sqrt{2} - \frac{15 - 42 + 35}{105} = \frac{22}{105}\sqrt{2} - \frac{8}{105}$

38. $\int \csc^4 x \cot^6 x\, dx = \int \cot^6 x (\cot^2 x + 1) \csc^2 x\, dx = \int u^6 (u^2 + 1) \cdot (-du)$ $[u = \cot x, du = -\csc^2 x\, dx]$

$= \int u^6(u^2 + 1) \cdot (-du)$ $[u = \cot x, du = -\csc^2 x\, dx]$

$= \int (-u^8 - u^6)\, du = -\frac{1}{9}u^9 - \frac{1}{7}u^7 + C = -\frac{1}{9}\cot^9 x - \frac{1}{7}\cot^7 x + C$

39. $\displaystyle I = \int \csc x\, dx = \int \frac{\csc x (\csc x - \cot x)}{\csc x - \cot x}\, dx = \int \frac{-\csc x \cot x + \csc^2 x}{\csc x - \cot x}\, dx$. Let $u = \csc x - \cot x \Rightarrow$

$du = (-\csc x \cot x + \csc^2 x)\, dx$. Then $I = \int du/u = \ln|u| = \ln|\csc x - \cot x| + C$.

40. Let $u = \csc x$, $dv = \csc^2 x\, dx$. Then $du = -\csc x\, \cot x\, dx$, $v = -\cot x \quad \Rightarrow$

$$\int \csc^3 x\, dx = -\csc x\, \cot x - \int \csc x\, \cot^2 x\, dx = -\csc x\, \cot x - \int \csc x\,(\csc^2 x - 1)\, dx$$
$$= -\csc x\, \cot x + \int \csc x\, dx - \int \csc^3 x\, dx$$

Solving for $\int \csc^3 x\, dx$ and using Exercise 39, we get

$\int \csc^3 x\, dx = -\frac{1}{2}\csc x\, \cot x + \frac{1}{2}\int \csc x\, dx = -\frac{1}{2}\csc x\, \cot x + \frac{1}{2}\ln|\csc x - \cot x| + C$. Thus,

$$\int_{\pi/6}^{\pi/3} \csc^3 x\, dx = \left[-\tfrac{1}{2}\csc x\, \cot x + \tfrac{1}{2}\ln|\csc x - \cot x|\right]_{\pi/6}^{\pi/3}$$
$$= -\tfrac{1}{2}\cdot\tfrac{2}{\sqrt{3}}\cdot\tfrac{1}{\sqrt{3}} + \tfrac{1}{2}\ln\left|\tfrac{2}{\sqrt{3}} - \tfrac{1}{\sqrt{3}}\right| + \tfrac{1}{2}\cdot 2\cdot\sqrt{3} - \tfrac{1}{2}\ln\left|2 - \sqrt{3}\right|$$
$$= -\tfrac{1}{3} + \sqrt{3} + \tfrac{1}{2}\ln\tfrac{1}{\sqrt{3}} - \tfrac{1}{2}\ln\left(2 - \sqrt{3}\right) \approx 1.7825$$

41. $\int \sin 8x\, \cos 5x\, dx \overset{2a}{=} \int \frac{1}{2}[\sin(8x - 5x) + \sin(8x + 5x)]\, dx = \frac{1}{2}\int \sin 3x\, dx + \frac{1}{2}\int \sin 13x\, dx$

$$= -\tfrac{1}{6}\cos 3x - \tfrac{1}{26}\cos 13x + C$$

42. $\int \cos \pi x\, \cos 4\pi x\, dx \overset{2c}{=} \int \frac{1}{2}[\cos(\pi x - 4\pi x) + \cos(\pi x + 4\pi x)]\, dx = \frac{1}{2}\int \cos(-3\pi x)\, dx + \frac{1}{2}\int \cos(5\pi x)\, dx$

$$= \tfrac{1}{2}\int \cos 3\pi x\, dx + \tfrac{1}{2}\int \cos 5\pi x\, dx = \tfrac{1}{6\pi}\sin 3\pi x + \tfrac{1}{10\pi}\sin 5\pi x + C$$

43. $\int \sin 5\theta\, \sin\theta\, d\theta \overset{2b}{=} \int \frac{1}{2}[\cos(5\theta - \theta) - \cos(5\theta + \theta)]\, d\theta = \frac{1}{2}\int \cos 4\theta\, d\theta - \frac{1}{2}\int \cos 6\theta\, d\theta = \frac{1}{8}\sin 4\theta - \frac{1}{12}\sin 6\theta + C$

44. $\displaystyle\int \frac{\cos x + \sin x}{\sin 2x}\, dx = \frac{1}{2}\int \frac{\cos x + \sin x}{\sin x\, \cos x}\, dx = \frac{1}{2}\int (\csc x + \sec x)\, dx$

$$= \tfrac{1}{2}\Big(\ln|\csc x - \cot x| + \ln|\sec x + \tan x|\Big) + C \qquad \text{[by Exercise 39 and (1)]}$$

45. $\int_0^{\pi/6} \sqrt{1 + \cos 2x}\, dx = \int_0^{\pi/6} \sqrt{1 + (2\cos^2 x - 1)}\, dx = \int_0^{\pi/6} \sqrt{2\cos^2 x}\, dx = \sqrt{2}\int_0^{\pi/6} \sqrt{\cos^2 x}\, dx$

$$= \sqrt{2}\int_0^{\pi/6} |\cos x|\, dx = \sqrt{2}\int_0^{\pi/6} \cos x\, dx \qquad [\text{since } \cos x > 0 \text{ for } 0 \le x \le \pi/6]$$
$$= \sqrt{2}\left[\sin x\right]_0^{\pi/6} = \sqrt{2}\left(\tfrac{1}{2} - 0\right) = \tfrac{1}{2}\sqrt{2}$$

46. $\int_0^{\pi/4} \sqrt{1 - \cos 4\theta}\, d\theta = \int_0^{\pi/4} \sqrt{1 - (1 - 2\sin^2(2\theta))}\, d\theta = \int_0^{\pi/4} \sqrt{2\sin^2(2\theta)}\, d\theta = \sqrt{2}\int_0^{\pi/4} \sqrt{\sin^2(2\theta)}\, d\theta$

$$= \sqrt{2}\int_0^{\pi/4} |\sin 2\theta|\, d\theta = \sqrt{2}\int_0^{\pi/4} \sin 2\theta\, d\theta \qquad [\text{since } \sin 2\theta \ge 0 \text{ for } 0 \le \theta \le \pi/4]$$
$$= \sqrt{2}\left[-\tfrac{1}{2}\cos 2\theta\right]_0^{\pi/4} = -\tfrac{1}{2}\sqrt{2}\,(0 - 1) = \tfrac{1}{2}\sqrt{2}$$

47. $\displaystyle\int \frac{1 - \tan^2 x}{\sec^2 x}\, dx = \int \left(\cos^2 x - \sin^2 x\right) dx = \int \cos 2x\, dx = \tfrac{1}{2}\sin 2x + C$

48. $\displaystyle\int \frac{dx}{\cos x - 1} = \int \frac{1}{\cos x - 1} \cdot \frac{\cos x + 1}{\cos x + 1}\, dx = \int \frac{\cos x + 1}{\cos^2 x - 1}\, dx = \int \frac{\cos x + 1}{-\sin^2 x}\, dx$

$= \int \left(-\cot x \ \csc x - \csc^2 x\right) dx = \csc x + \cot x + C$

49. $\int x \tan^2 x\, dx = \int x(\sec^2 x - 1)\, dx = \int x \sec^2 x\, dx - \int x\, dx$

$= x \tan x - \int \tan x\, dx - \frac{1}{2}x^2 \qquad \begin{bmatrix} u = x, & dv = \sec^2 x\, dx \\ du = dx, & v = \tan x \end{bmatrix}$

$= x \tan x - \ln|\sec x| - \frac{1}{2}x^2 + C$