1. Let $u = \ln x$, $dv = x^2\,dx \quad \Rightarrow \quad du = \frac{1}{x}\,dx$, $v = \frac{1}{3}x^3$. Then by Equation 2,

$$\int x^2 \ln x\,dx = (\ln x)\left(\tfrac{1}{3}x^3\right) - \int \left(\tfrac{1}{3}x^3\right)\left(\tfrac{1}{x}\right)\,dx = \tfrac{1}{3}x^3\ \ln x - \tfrac{1}{3}\int x^2\,dx = \tfrac{1}{3}x^3\ \ln x - \tfrac{1}{3}\left(\tfrac{1}{3}x^3\right) + C$$
$$= \tfrac{1}{3}x^3\ \ln x - \tfrac{1}{9}x^3 + C \quad \left[\text{or } \tfrac{1}{3}x^3\left(\ln x - \tfrac{1}{3}\right) + C\right]$$

2. Let $u = \theta$, $dv = \cos\theta\,d\theta \quad \Rightarrow \quad du = d\theta$, $v = \sin\theta$. Then by Equation 2,

$$\int \theta\cos\theta\,d\theta = \theta\sin\theta - \int \sin\theta\,d\theta = \theta\sin\theta + \cos\theta + C.$$

Note: A mnemonic device which is helpful for selecting u when using integration by parts is the LIATE principle of precedence for u:

Logarithmic
Inverse trigonometric
Algebraic
Trigonometric
Exponential

If the integrand has several factors, then we try to choose among them a u which appears as high as possible on the list. For example, in $\int xe^{2x}\,dx$ the integrand is xe^{2x}, which is the product of an algebraic function (x) and an exponential function (e^{2x}). Since Algebraic appears before Exponential, we choose $u = x$. Sometimes the integration turns out to be similar regardless of the selection of u and dv, but it is advisable to refer to LIATE when in doubt.

3. Let $u = x$, $dv = \cos 5x\,dx \quad \Rightarrow \quad du = dx$, $v = \frac{1}{5}\sin 5x$. Then by Equation 2,

$$\int x\cos 5x\,dx = \tfrac{1}{5}x\sin 5x - \int \tfrac{1}{5}\sin 5x\,dx = \tfrac{1}{5}x\sin 5x + \tfrac{1}{25}\cos 5x + C.$$

4. Let $u = y$, $dv = e^{0.2y}dy \quad \Rightarrow \quad du = dy$, $v = \frac{1}{0.2}e^{0.2y}$. Then by Equation 2,

$$\int ye^{0.2y}dy = 5ye^{0.2y} - \int 5e^{0.2y}dy = 5ye^{0.2y} - 25e^{0.2y} + C.$$

5. Let $u = t$, $dv = e^{-3t}dt \quad \Rightarrow \quad du = dt$, $v = -\frac{1}{3}e^{-3t}$. Then by Equation 2,

$$\int te^{-3t}dt = -\tfrac{1}{3}te^{-3t} - \int -\tfrac{1}{3}e^{-3t}dt = -\tfrac{1}{3}te^{-3t} + \tfrac{1}{3}\int e^{-3t}dt = -\tfrac{1}{3}te^{-3t} - \tfrac{1}{9}e^{-3t} + C.$$

6. Let $u = x - 1$, $dv = \sin\pi x\,dx \quad \Rightarrow \quad du = dx$, $v = -\frac{1}{\pi}\cos\pi x$. Then by Equation 2,

$$\int (x-1)\sin\pi x\,dx = -\frac{1}{\pi}(x-1)\cos\pi x - \int -\frac{1}{\pi}\cos\pi x\,dx = -\frac{1}{\pi}(x-1)\cos\pi x + \frac{1}{\pi}\int \cos\pi x\,dx$$
$$= -\frac{1}{\pi}(x-1)\cos\pi x + \frac{1}{\pi^2}\sin\pi x + C$$

7.

First let $u = x^2 + 2x$, $dv = \cos x\,dx \quad \Rightarrow \quad du = (2x+2)\,dx$, $v = \sin x$. Then by Equation 2,

$I = \int (x^2+2x)\cos x\,dx = (x^2+2x)\sin x - \int (2x+2)\sin x\,dx$. Next let $U = 2x+2$, $dV = \sin x\,dx \quad \Rightarrow \quad dU = 2\,dx$, $V = -\cos x$, so $\int (2x+2)\sin x\,dx = -(2x+2)\cos x - \int -2\cos x\,dx = -(2x+2)\cos x + 2\sin x$. Thus,

$I = (x^2+2x)\sin x + (2x+2)\cos x - 2\sin x + C.$

8. First let $u = t^2$, $dv = \sin \beta t\, dt \quad \Rightarrow \quad du = 2t\, dt$, $v = -\frac{1}{\beta}\cos \beta t$. Then by Equation 2,

$I = \int t^2 \sin \beta t\, dt = -\frac{1}{\beta}t^2 \cos \beta t - \int -\frac{2}{\beta} t \cos \beta t\, dt$. Next let $U = t$, $dV = \cos \beta t\, dt \quad \Rightarrow \quad dU = dt$,

$V = \dfrac{1}{\beta}\sin \beta t$, so $\int t \cos \beta t\, dt = \dfrac{1}{\beta} t \sin \beta t - \displaystyle\int \dfrac{1}{\beta} \sin \beta t\, dt = \dfrac{1}{\beta} t \sin \beta t + \dfrac{1}{\beta^2}\cos \beta t$. Thus,

$$I = -\frac{1}{\beta}t^2 \cos \beta t + \frac{2}{\beta}\left(\frac{1}{\beta} t \sin \beta t + \frac{1}{\beta^2}\cos \beta t\right) + C = -\frac{1}{\beta}t^2 \cos \beta t + \frac{2}{\beta^2} t \sin \beta t + \frac{2}{\beta^3}\cos \beta t + C.$$

9. Let $u = \ln \sqrt[3]{x}$, $dv = dx \quad \Rightarrow \quad du = \dfrac{1}{\sqrt[3]{x}}\left(\dfrac{1}{3}x^{-2/3}\right) dx = \dfrac{1}{3x}\, dx$, $v = x$. Then

$$\int \ln \sqrt[3]{x}\, dx = x \ln \sqrt[3]{x} - \int x \cdot \frac{1}{3x} dx = x \ln \sqrt[3]{x} - \frac{1}{3}x + C.$$

Second solution: Rewrite $\int \ln \sqrt[3]{x}\, dx = \frac{1}{3}\int \ln x\, dx$, and apply Example 2.

Third solution: Substitute $y = \sqrt[3]{x}$, to obtain $\int \ln \sqrt[3]{x}\, dx = 3\int y^2 \ln y\, dy$, and apply Exercise 1.

10. Let $u = \sin^{-1} x$, $dv = dx \quad \Rightarrow \quad du = \dfrac{dx}{\sqrt{1-x^2}}$, $v = x$. Then $\int \sin^{-1} x\, dx = x \sin^{-1} x - \displaystyle\int \dfrac{x}{\sqrt{1-x^2}}\, dx$. Setting

$t = 1 - x^2$, we get $dt = -2x\, dx$, so $-\displaystyle\int \dfrac{x\, dx}{\sqrt{1-x^2}} = -\displaystyle\int t^{-1/2}\left(-\tfrac{1}{2}\, dt\right) = \tfrac{1}{2}(2t^{1/2}) + C = t^{1/2} + C = \sqrt{1-x^2} + C$.

Hence, $\int \sin^{-1} x\, dx = x \sin^{-1} x + \sqrt{1-x^2} + C$.

11. Let $u = \arctan 4t$, $dv = dt \quad \Rightarrow \quad du = \dfrac{4}{1+(4t)^2}\, dt = \dfrac{4}{1+16t^2}\, dt$, $v = t$. Then

$$\int \arctan 4t\, dt = t \arctan 4t - \int \frac{4t}{1+16t^2}\, dt = t \arctan 4t - \frac{1}{8}\int \frac{32t}{1+16t^2}\, dt = t \arctan 4t - \tfrac{1}{8}\ln(1+16t^2) + C.$$

12. Let $u = \ln p$, $dv = p^5 dp \quad \Rightarrow \quad du = \dfrac{1}{p}\, dp$, $v = \frac{1}{6}p^6$. Then $\int p^5 \ln p\, dp = \frac{1}{6}p^6 \ln p - \frac{1}{6}\int p^5 dp = \frac{1}{6}p^6 \ln p - \frac{1}{36}p^6 + C$.

13. Let $u = t$, $dv = \sec^2 2t\, dt \quad \Rightarrow \quad du = dt$, $v = \frac{1}{2}\tan 2t$. Then

$\int t \sec^2 2t\, dt = \frac{1}{2} t \tan 2t - \frac{1}{2}\int \tan 2t\, dt = \frac{1}{2} t \tan 2t - \frac{1}{4}\ln|\sec 2t| + C$.

14. Let $u = s$, $dv = 2^s\, ds \quad \Rightarrow \quad du = ds$, $v = \dfrac{1}{\ln 2}2^s$. Then

$$\int s\, 2^s\, ds = \frac{1}{\ln 2}s\, 2^s - \frac{1}{\ln 2}\int 2^s\, ds = \frac{1}{\ln 2}s\, 2^s - \frac{1}{(\ln 2)^2}2^s + C \quad \left[\text{or } \frac{2^s}{(\ln 2)^2}(s \ln 2 - 1) + C\right].$$

15. First let $u = (\ln x)^2$, $dv = dx \quad \Rightarrow \quad du = 2\ln x \cdot \frac{1}{x}\,dx$, $v = x$. Then by Equation 2,

$I = \int (\ln x)^2\,dx = x(\ln x)^2 - 2\int x\ln x \cdot \frac{1}{x}\,dx = x(\ln x)^2 - 2\int \ln x\,dx$. Next let $U = \ln x$, $dV = dx \quad \Rightarrow$

$dU = 1/x\,dx$, $V = x$ to get $\int \ln x\,dx = x\ln x - \int x\cdot(1/x)\,dx = x\ln x - \int dx = x\ln x - x + C_1$. Thus,

$I = x(\ln x)^2 - 2(x\ln x - x + C_1) = x(\ln x)^2 - 2x\ln x + 2x + C$, where $C = -2C_1$.

16. Let $u = t$, $dv = \sinh mt\,dt \quad \Rightarrow \quad du = dt$, $v = \dfrac{1}{m}\cosh mt$. Then

$$\int t\sinh mt\,dt = \frac{1}{m}t\cosh mt - \int \frac{1}{m}\cosh mt\,dt = \frac{1}{m}t\cosh mt - \frac{1}{m^2}\sinh mt + C \;\; [m \neq 0].$$

17. First let $u = \sin 3\theta$, $dv = e^{2\theta}\,d\theta \quad \Rightarrow \quad du = 3\cos 3\theta\,d\theta$, $v = \frac{1}{2}e^{2\theta}$. Then

$I = \int e^{2\theta}\sin 3\theta\,d\theta = \frac{1}{2}e^{2\theta}\sin 3\theta - \frac{3}{2}\int e^{2\theta}\cos 3\theta\,d\theta$. Next let $U = \cos 3\theta$, $dV = e^{2\theta}\,d\theta \quad \Rightarrow \quad dU = -3\sin 3\theta\,d\theta$,

$V = \frac{1}{2}e^{2\theta}$ to get $\int e^{2\theta}\cos 3\theta\,d\theta = \frac{1}{2}e^{2\theta}\cos 3\theta + \frac{3}{2}\int e^{2\theta}\sin 3\theta\,d\theta$. Substituting in the previous formula gives

$I = \frac{1}{2}e^{2\theta}\sin 3\theta - \frac{3}{4}e^{2\theta}\cos 3\theta - \frac{9}{4}\int e^{2\theta}\sin 3\theta\,d\theta = \frac{1}{2}e^{2\theta}\sin 3\theta - \frac{3}{4}e^{2\theta}\cos 3\theta - \frac{9}{4}I \quad \Rightarrow$

$\frac{13}{4}I = \frac{1}{2}e^{2\theta}\sin 3\theta - \frac{3}{4}e^{2\theta}\cos 3\theta + C_1$. Hence, $I = \frac{1}{13}e^{2\theta}(2\sin 3\theta - 3\cos 3\theta) + C$, where $C = \frac{4}{13}C_1$.

18.

First let $u = e^{-\theta}$, $dv = \cos 2\theta\,d\theta \quad \Rightarrow \quad du = -e^{-\theta}\,d\theta$, $v = \frac{1}{2}\sin 2\theta$. Then

$I = \int e^{-\theta}\cos 2\theta\,d\theta = \frac{1}{2}e^{-\theta}\sin 2\theta - \int \frac{1}{2}\sin 2\theta\left(-e^{-\theta}\,d\theta\right) = \frac{1}{2}e^{-\theta}\sin 2\theta + \frac{1}{2}\int e^{-\theta}\sin 2\theta\,d\theta$.

Next let $U = e^{-\theta}$, $dV = \sin 2\theta\,d\theta \quad \Rightarrow \quad dU = -e^{-\theta}\,d\theta$, $V = -\frac{1}{2}\cos 2\theta$, so

$\int e^{-\theta}\sin 2\theta\,d\theta = -\frac{1}{2}e^{-\theta}\cos 2\theta - \int\left(-\frac{1}{2}\right)\cos 2\theta\left(-e^{-\theta}\,d\theta\right) = -\frac{1}{2}e^{-\theta}\cos 2\theta - \frac{1}{2}\int e^{-\theta}\cos 2\theta\,d\theta$.

So $I = \frac{1}{2}e^{-\theta}\sin 2\theta + \frac{1}{2}\left[\left(-\frac{1}{2}e^{-\theta}\cos 2\theta\right) - \frac{1}{2}I\right] = \frac{1}{2}e^{-\theta}\sin 2\theta - \frac{1}{4}e^{-\theta}\cos 2\theta - \frac{1}{4}I \quad \Rightarrow$

$\frac{5}{4}I = \frac{1}{2}e^{-\theta}\sin 2\theta - \frac{1}{4}e^{-\theta}\cos 2\theta + C_1 \quad \Rightarrow \quad I = \frac{4}{5}\left(\frac{1}{2}e^{-\theta}\sin 2\theta - \frac{1}{4}e^{-\theta}\cos 2\theta + C_1\right) = \frac{2}{5}e^{-\theta}\sin 2\theta - \frac{1}{5}e^{-\theta}\cos 2\theta + C$.

19. First let $u = z^3$, $dv = e^z dz \quad \Rightarrow \quad du = 3z^2 dz$, $v = e^z$. Then $I_1 = \int z^3 e^z dz = z^3 e^z - 3\int z^2 e^z dz$. Next let $u_1 = z^2$,

$dv_1 = e^z dz \quad \Rightarrow \quad du_1 = 2z\,dz$, $v_1 = e^z$. Then $I_2 = z^2 e^z - 2\int z e^z dz$. Finally, let $u_2 = z$, $dv_2 = e^z dz \quad \Rightarrow \quad du_2 = dz$,

$v_2 = e^z$. Then $\int z e^z dz = z e^z - \int e^z dz = z e^z - e^z + C_1$. Substituting in the expression for I_2, we get

$I_2 = z^2 e^z - 2(z e^z - e^z + C_1) = z^2 e^z - 2z e^z + 2e^z - 2C_1$. Substituting the last expression for I_2 into I_1 gives

$I_1 = z^3 e^z - 3(z^2 e^z - 2z e^z + 2e^z - 2C_1) = z^3 e^z - 3z^2 e^z + 6z e^z - 6e^z + C$, where $C = 6C_1$.

20. $\int x\tan^2 x\,dx = \int x(\sec^2 x - 1)\,dx = \int x\sec^2 x\,dx - \int x\,dx$. Let $u = x$, $dv = \sec^2 x\,dx \quad \Rightarrow \quad du = dx$, $v = \tan x$.

Then by Equation 2, $\int x\sec^2 x\,dx = x\tan x - \int \tan x\,dx = x\tan x - \ln|\sec x|$, and thus,

$\int x\tan^2 x\,dx = x\tan x - \ln|\sec x| - \frac{1}{2}x^2 + C$.

21. Let $u = xe^{2x}$, $dv = \dfrac{1}{(1+2x)^2}\,dx \quad\Rightarrow\quad du = (x \cdot 2e^{2x} + e^{2x} \cdot 1)\,dx = e^{2x}(2x+1)\,dx$, $v = -\dfrac{1}{2(1+2x)}$.

Then by Equation 2,

$$\int \frac{xe^{2x}}{(1+2x)^2}\,dx = -\frac{xe^{2x}}{2(1+2x)} + \frac{1}{2}\int \frac{e^{2x}(2x+1)}{1+2x}\,dx = -\frac{xe^{2x}}{2(1+2x)} + \frac{1}{2}\int e^{2x}dx = -\frac{xe^{2x}}{2(1+2x)} + \frac{1}{4}e^{2x} + C$$

The answer could be written as $\dfrac{e^{2x}}{4(2x+1)} + C$.

22.

First let $u = (\arcsin x)^2$, $dv = dx \quad\Rightarrow\quad du = 2\arcsin x \cdot \dfrac{1}{\sqrt{1-x^2}}\,dx$, $v = x$. Then

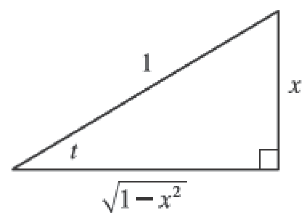
$I = \displaystyle\int (\arcsin x)^2 dx = x(\arcsin x)^2 - 2\int \frac{x \arcsin x}{\sqrt{1-x^2}}\,dx$. To simplify the last integral, let $t = \arcsin x$ $[x = \sin t]$, so

$dt = \dfrac{1}{\sqrt{1-x^2}}\,dx$, and $\displaystyle\int \frac{x\arcsin x}{\sqrt{1-x^2}}\,dx = \int t \sin t\,dt$. To evaluate just the last integral, now let $U = t$, $dV = \sin t\,dt \quad\Rightarrow$

$dU = dt$, $V = -\cos t$. Thus,

$$\int t \sin t\,dt = -t\cos t + \int \cos t\,dt = -t\cos t + \sin t + C$$

$$= -\arcsin x \cdot \frac{\sqrt{1-x^2}}{1} + x + C_1 \quad \text{[refer to the figure]}$$



Returning to I, we get $I = x(\arcsin x)^2 + 2\sqrt{1-x^2}\,\arcsin x - 2x + C$,

where $C = -2C_1$.

23. Let $u = x$, $dv = \cos \pi x\,dx \quad\Rightarrow\quad du = dx$, $v = \frac{1}{\pi}\sin \pi x$. Then

$$\int_0^{1/2} x\cos \pi x\,dx = \left[\frac{1}{\pi}x \sin \pi x\right]_0^{1/2} - \int_0^{1/2} \frac{1}{\pi}\sin \pi x\,dx = \frac{1}{2\pi} - 0 - \frac{1}{\pi}\left[-\frac{1}{\pi}\cos \pi x\right]_0^{1/2}$$

$$= \frac{1}{2\pi} + \frac{1}{\pi^2}(0-1) = \frac{1}{2\pi} - \frac{1}{\pi^2} \quad\text{or}\quad \frac{\pi-2}{2\pi^2}$$

24. First let $u = x^2+1$, $dv = e^{-x}\,dx \quad\Rightarrow\quad du = 2x\,dx$, $v = -e^{-x}$. By (6),

$\int_0^1 (x^2+1)e^{-x}\,dx = \left[-(x^2+1)e^{-x}\right]_0^1 + \int_0^1 2xe^{-x}\,dx = -2e^{-1} + 1 + 2\int_0^1 xe^{-x}\,dx$.

Next let $U = x$, $dV = e^{-x}\,dx \quad\Rightarrow\quad dU = dx$, $V = -e^{-x}$. By (6) again,

$\int_0^1 xe^{-x}\,dx = \left[-xe^{-x}\right]_0^1 + \int_0^1 e^{-x}\,dx = -e^{-1} + \left[-e^{-x}\right]_0^1 = -e^{-1} - e^{-1} + 1 = -2e^{-1} + 1$. So

$\int_0^1 (x^2+1)e^{-x}\,dx = -2e^{-1} + 1 + 2(-2e^{-1}+1) = -2e^{-1} + 1 - 4e^{-1} + 2 = -6e^{-1} + 3$.

25. Let $u = t$, $dv = \cosh t\,dt \quad\Rightarrow\quad du = dt$, $v = \sinh t$. Then

$\int_0^1 t\cosh t\,dt = \left[t\sinh t\right]_0^1 - \int_0^1 \sinh t\,dt = (\sinh 1 - \sinh 0) - \left[\cosh t\right]_0^1 = \sinh 1 - (\cosh 1 - \cosh 0)$

$= \sinh 1 - \cosh 1 + 1.$

We can use the definitions of sinh and cosh to write the answer in terms of e:

$\sinh 1 - \cosh 1 + 1 = \frac{1}{2}(e^1 - e^{-1}) - \frac{1}{2}(e^1 + e^{-1}) + 1 = -e^{-1} + 1 = 1 - 1/e.$

26. Let $u = \ln y$, $dv = \dfrac{1}{\sqrt{y}}\,dy = y^{-1/2}\,dy \quad\Rightarrow\quad du = \dfrac{1}{y}\,dy$, $v = 2y^{1/2}$. Then

$$\int_4^9 \frac{\ln y}{\sqrt{y}}\,dy = \left[2\sqrt{y}\,\ln y\right]_4^9 - \int_4^9 2y^{-1/2}\,dy = (6\ln 9 - 4\ln 4) - \left[4\sqrt{y}\right]_4^9 = 6\ln 9 - 4\ln 4 - (12 - 8)$$

$$= 6\ln 9 - 4\ln 4 - 4$$

27. Let $u = \ln r$, $dv = r^3 dr \quad\Rightarrow\quad du = \frac{1}{r}\,dr$, $v = \frac{1}{4}r^4$. Then

$\int_1^3 r^3\ln r\,dr = \left[\frac{1}{4}r^4\ln r\right]_1^3 - \int_1^3 \frac{1}{4}r^3 dr = \frac{81}{4}\ln 3 - 0 - \frac{1}{4}\left[\frac{1}{4}r^4\right]_1^3 = \frac{81}{4}\ln 3 - \frac{1}{16}(81 - 1) = \frac{81}{4}\ln 3 - 5.$

28. First let $u = t^2$, $dv = \sin 2t\,dt \quad\Rightarrow\quad du = 2t\,dt$, $v = -\frac{1}{2}\cos 2t$. By (6),

$\int_0^{2\pi} t^2\sin 2t\,dt = \left[-\frac{1}{2}t^2\cos 2t\right]_0^{2\pi} + \int_0^{2\pi} t\cos 2t\,dt = -2\pi^2 + \int_0^{2\pi} t\cos 2t\,dt$. Next let $U = t$, $dV = \cos 2t\,dt \quad\Rightarrow$ $dU = dt$, $V = \frac{1}{2}\sin 2t$. By (6) again,

$\int_0^{2\pi} t\cos 2t\,dt = \left[\frac{1}{2}t\sin 2t\right]_0^{2\pi} - \int_0^{2\pi}\frac{1}{2}\sin 2t\,dt = 0 - \left[-\frac{1}{4}\cos 2t\right]_0^{2\pi} = \frac{1}{4} - \frac{1}{4} = 0$. Thus, $\int_0^{2\pi} t^2\sin 2t\,dt = -2\pi^2$.

29. Let $u = y$, $dv = \dfrac{dy}{e^{2y}} = e^{-2y}dy \quad\Rightarrow\quad du = dy$, $v = -\frac{1}{2}e^{-2y}$. Then

$$\int_0^1 \frac{y}{e^{2y}}\,dy = \left[-\tfrac{1}{2}ye^{-2y}\right]_0^1 + \tfrac{1}{2}\int_0^1 e^{-2y}dy = \left(-\tfrac{1}{2}e^{-2} + 0\right) - \tfrac{1}{4}\left[e^{-2y}\right]_0^1 = -\tfrac{1}{2}e^{-2} - \tfrac{1}{4}e^{-2} + \tfrac{1}{4} = \tfrac{1}{4} - \tfrac{3}{4}e^{-2}.$$

30. Let $u = \arctan(1/x)$, $dv = dx \quad\Rightarrow\quad du = \dfrac{1}{1 + (1/x)^2}\cdot\dfrac{-1}{x^2}\,dx = \dfrac{-dx}{x^2 + 1}$, $v = x$. Then

$$\int_1^{\sqrt{3}} \arctan\left(\frac{1}{x}\right)\,dx = \left[x\arctan\left(\frac{1}{x}\right)\right]_1^{\sqrt{3}} + \int_1^{\sqrt{3}} \frac{x\,dx}{x^2 + 1} = \sqrt{3}\,\frac{\pi}{6} - 1\cdot\frac{\pi}{4} + \frac{1}{2}\left[\ln(x^2 + 1)\right]_1^{\sqrt{3}}$$

$$= \frac{\pi\sqrt{3}}{6} - \frac{\pi}{4} + \frac{1}{2}(\ln 4 - \ln 2) = \frac{\pi\sqrt{3}}{6} - \frac{\pi}{4} + \frac{1}{2}\ln\frac{4}{2} = \frac{\pi\sqrt{3}}{6} - \frac{\pi}{4} + \frac{1}{2}\ln 2$$

31. Let $u = \cos^{-1} x$, $dv = dx \quad \Rightarrow \quad du = -\dfrac{dx}{\sqrt{1-x^2}}$, $v = x$. Then

$I = \displaystyle\int_0^{1/2} \cos^{-1} x\,dx = \left[x \cos^{-1} x\right]_0^{1/2} + \int_0^{1/2} \tfrac{x\,dx}{\sqrt{1-x^2}} = \tfrac{1}{2} \cdot \tfrac{\pi}{3} + \int_1^{3/4} t^{-1/2}\left[-\tfrac{1}{2}dt\right]$, where $t = 1 - x^2 \quad \Rightarrow$

$dt = -2x\,dx$. Thus, $I = \frac{\pi}{6} + \frac{1}{2}\int_{3/4}^{1} t^{-1/2}\,dt = \frac{\pi}{6} + \left[\sqrt{t}\right]_{3/4}^{1} = \frac{\pi}{6} + 1 - \frac{\sqrt{3}}{2} = \frac{1}{6}\left(\pi + 6 - 3\sqrt{3}\right)$.

32. Let $u = (\ln x)^2$, $dv = x^{-3}\,dx \quad \Rightarrow \quad du = \dfrac{2\ln x}{x}\,dx$, $v = -\frac{1}{2}x^{-2}$. Then

$I = \displaystyle\int_1^2 \frac{(\ln x)^2}{x^3}\,dx = \left[-\frac{(\ln x)^2}{2x^2}\right]_1^2 + \int_1^2 \frac{\ln x}{x^3}\,dx$. Now let $U = \ln x$, $dV = x^{-3}\,dx \quad \Rightarrow \quad dU = \dfrac{1}{x}\,dx$, $V = -\frac{1}{2}x^{-2}$.

Then

$$\int_1^2 \frac{\ln x}{x^3}\,dx = \left[-\frac{\ln x}{2x^2}\right]_1^2 + \tfrac{1}{2}\int_1^2 x^{-3}dx = -\tfrac{1}{8}\ln 2 + 0 + \tfrac{1}{2}\left[-\tfrac{1}{2x^2}\right]_1^2 = -\tfrac{1}{8}\ln 2 + \tfrac{1}{2}\left(-\tfrac{1}{8} + \tfrac{1}{2}\right) = \tfrac{3}{16} - \tfrac{1}{8}\ln 2.$$

Thus $I = \left(-\frac{1}{8}(\ln 2)^2 + 0\right) + \left(\frac{3}{16} - \frac{1}{8}\ln 2\right) = -\frac{1}{8}(\ln 2)^2 - \frac{1}{8}\ln 2 + \frac{3}{16}$.

33. Let $u = \ln(\sin x)$, $dv = \cos x\,dx \quad \Rightarrow \quad du = \dfrac{\cos x}{\sin x}\,dx$, $v = \sin x$. Then

$I = \int \cos x \ln(\sin x)\,dx = \sin x \ln(\sin x) - \int \cos x\,dx = \sin x \ln(\sin x) - \sin x + C$.

Another method: Substitute $t = \sin x$, so $dt = \cos x\,dx$. Then $I = \int \ln t\,dt = t\ln t - t + C$ (see Example 2) and so $I = \sin x\,(\ln \sin x - 1) + C$.

34. Let $u = r^2$, $dv = \dfrac{r}{\sqrt{4+r^2}}\,dr \quad \Rightarrow \quad du = 2r\,dr$, $v = \sqrt{4+r^2}$. By (6),

$$\begin{aligned}\int_0^1 \frac{r^3}{\sqrt{4+r^2}}\,dr &= \left[r^2\sqrt{4+r^2}\right]_0^1 - 2\int_0^1 r\sqrt{4+r^2}\,dr = \sqrt{5} - \tfrac{2}{3}\left[(4+r^2)^{3/2}\right]_0^1 \\ &= \sqrt{5} - \tfrac{2}{3}(5)^{3/2} + \tfrac{2}{3}(8) = \sqrt{5}\left(1 - \tfrac{10}{3}\right) + \tfrac{16}{3} = \tfrac{16}{3} - \tfrac{7}{3}\sqrt{5}\end{aligned}$$

35. Let $u = (\ln x)^2$, $dv = x^4\,dx \quad \Rightarrow \quad du = 2\dfrac{\ln x}{x}\,dx$, $v = \dfrac{x^5}{5}$. By (6),

$$\int_1^2 x^4(\ln x)^2\,dx = \left[\frac{x^5}{5}(\ln x)^2\right]_1^2 - 2\int_1^2 \frac{x^4}{5}\ln x\,dx = \tfrac{32}{5}(\ln 2)^2 - 0 - 2\int_1^2 \frac{x^4}{5}\ln x\,dx.$$

Let $U = \ln x$, $dV = \dfrac{x^4}{5}\,dx \quad \Rightarrow \quad dU = \dfrac{1}{x}\,dx$, $V = \dfrac{x^5}{25}$.

Then $\displaystyle\int_1^2 \frac{x^4}{5}\ln x\,dx = \left[\frac{x^5}{25}\ln x\right]_1^2 - \int_1^2 \frac{x^4}{25}\,dx = \tfrac{32}{25}\ln 2 - 0 - \left[\frac{x^5}{125}\right]_1^2 = \tfrac{32}{25}\ln 2 - \left(\tfrac{32}{125} - \tfrac{1}{125}\right)$.

So $\int_1^2 x^4(\ln x)^2\,dx = \frac{32}{5}(\ln 2)^2 - 2\left(\frac{32}{25}\ln 2 - \frac{31}{125}\right) = \frac{32}{5}(\ln 2)^2 - \frac{64}{25}\ln 2 + \frac{62}{125}$.

36.
Let $u = \sin(t-s)$, $dv = e^s\,ds \quad\Rightarrow\quad du = -\cos(t-s)\,ds$, $v = e^s$. Then

$I = \int_0^t e^s \sin(t-s)\,ds = \Big[e^s \sin(t-s)\Big]_0^t + \int_0^t e^s \cos(t-s)\,ds = e^t \sin 0 - e^0 \sin t + I_1$. For I_1, let $U = \cos(t-s)$,

$dV = e^s\,ds \quad\Rightarrow\quad dU = \sin(t-s)\,ds$, $V = e^s$. So $I_1 = \big[e^s \cos(t-s)\big]_0^t - \int_0^t e^s \sin(t-s)\,ds = e^t \cos 0 - e^0 \cos t - I$.

Thus, $I = -\sin t + e^t - \cos t - I \quad\Rightarrow\quad 2I = e^t - \cos t - \sin t \quad\Rightarrow\quad I = \frac{1}{2}(e^t - \cos t - \sin t)$.

37.
Let $y = \sqrt{x}$, so that $dy = \frac{1}{2}x^{-1/2}\,dx = \dfrac{1}{2\sqrt{x}}\,dx = \dfrac{1}{2y}\,dx$. Thus, $\int \cos\sqrt{x}\,dx = \int \cos y\,(2y\,dy) = 2\int y\cos y\,dy$. Now

use parts with $u = y$, $dv = \cos y\,dy$, $du = dy$, $v = \sin y$ to get $\int y\cos y\,dy = y\sin y - \int \sin y\,dy = y\sin y + \cos y + C_1$,

so $\int \cos\sqrt{x}\,dx = 2y\sin y + 2\cos y + C = 2\sqrt{x}\sin\sqrt{x} + 2\cos\sqrt{x} + C$.

38. Let $x = -t^2$, so that $dx = -2t\,dt$. Thus, $\int t^3 e^{-t^2}\,dt = \int (-t^2)e^{-t^2}\left(\frac{1}{2}\right)(-2t\,dt) = \frac{1}{2}\int xe^x\,dx$. Now use parts with

$u = x$, $dv = e^x\,dx$, $du = dx$, $v = e^x$ to get

$$\frac{1}{2}\int xe^x\,dx = \frac{1}{2}\left(xe^x - \int e^x\,dx\right) = \frac{1}{2}xe^x - \frac{1}{2}e^x + C = -\frac{1}{2}(1-x)e^x + C = -\frac{1}{2}(1+t^2)e^{-t^2} + C.$$

39.
Let $x = \theta^2$, so that $dx = 2\theta\,d\theta$. Thus, $\displaystyle\int_{\sqrt{\pi/2}}^{\sqrt{\pi}} \theta^3\cos(\theta^2)\;d\theta = \int_{\sqrt{\pi/2}}^{\sqrt{\pi}} \theta^2\cos(\theta^2)\cdot\frac{1}{2}(2\theta\,d\theta) = \frac{1}{2}\int_{\pi/2}^{\pi} x\cos x\,dx$. Now use

parts with $u = x$, $dv = \cos x\,dx$, $du = dx$, $v = \sin x$ to get

$$\begin{aligned}\frac{1}{2}\int_{\pi/2}^{\pi} x\cos x\,dx &= \frac{1}{2}\left(\big[x\sin x\big]_{\pi/2}^{\pi} - \int_{\pi/2}^{\pi}\sin x\,dx\right) = \frac{1}{2}\,\big[x\sin x + \cos x\big]_{\pi/2}^{\pi} \\ &= \frac{1}{2}(\pi\sin\pi + \cos\pi) - \frac{1}{2}\left(\frac{\pi}{2}\sin\frac{\pi}{2} + \cos\frac{\pi}{2}\right) = \frac{1}{2}(\pi\cdot 0 - 1) - \frac{1}{2}\left(\frac{\pi}{2}\cdot 1 + 0\right) = -\frac{1}{2} - \frac{\pi}{4}\end{aligned}$$

40. Let $x = \cos t$, so that $dx = -\sin t\,dt$. Thus,

$\int_0^\pi e^{\cos t}\sin 2t\,dt = \int_0^\pi e^{\cos t}(2\sin t\;\cos t)\,dt = \int_1^{-1} e^x\cdot 2x\,(-dx) = 2\int_{-1}^1 xe^x\,dx$. Now use parts with $u = x$,

$dv = e^x\,dx$, $du = dx$, $v = e^x$ to get

$$2\int_{-1}^1 xe^x\,dx = 2\left(\big[xe^x\big]_{-1}^1 - \int_{-1}^1 e^x\,dx\right) = 2\left(e^1 + e^{-1} - \big[\,e^x\,\big]_{-1}^1\right) = 2(e + e^{-1} - [e^1 - e^{-1}]) = 2(2e^{-1}) = 4/e.$$

41.

Let $y = 1 + x$, so that $dy = dx$. Thus, $\int x \ln(1+x)\,dx = \int (y-1)\ln y\,dy$. Now use parts with $u = \ln y$, $dv = (y-1)\,dy$, $du = \frac{1}{y}\,dy$, $v = \frac{1}{2}y^2 - y$ to get

$$\begin{aligned}\int (y-1)\ln y\,dy &= \left(\tfrac{1}{2}y^2 - y\right)\ln y - \int\left(\tfrac{1}{2}y - 1\right)dy = \tfrac{1}{2}y(y-2)\ln y - \tfrac{1}{4}y^2 + y + C \\ &= \tfrac{1}{2}(1+x)(x-1)\ln(1+x) - \tfrac{1}{4}(1+x)^2 + 1 + x + C,\end{aligned}$$

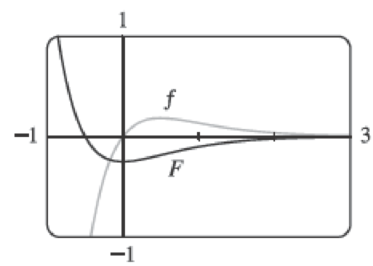
which can be written as $\frac{1}{2}(x^2-1)\ln(1+x) - \frac{1}{4}x^2 + \frac{1}{2}x + \frac{3}{4} + C$.

42. Let $y = \ln x$, so that $dy = \frac{1}{x}\,dx \quad \Rightarrow \quad dx = x\,dy = e^y\,dy$. Thus,

$\int \sin(\ln x)\,dx = \int \sin y\, e^y\,dy = \frac{1}{2}e^y(\sin y - \cos y) + C$ [by Example 4] $= \frac{1}{2}x[\sin(\ln x) - \cos(\ln x)] + C$.

43. Let $u = x$, $dv = e^{-2x}\,dx \quad \Rightarrow \quad du = dx$, $v = -\frac{1}{2}e^{-2x}$. Then

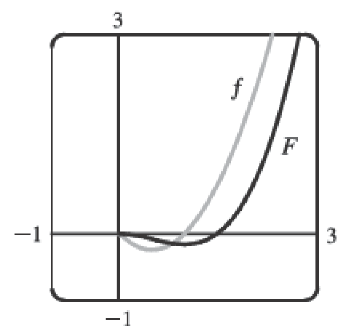
$\int xe^{-2x}\,dx = -\frac{1}{2}xe^{-2x} + \int \frac{1}{2}e^{-2x}\,dx = -\frac{1}{2}xe^{-2x} - \frac{1}{4}e^{-2x} + C$. We see from the graph that this is reasonable, since F has a minimum where f changes from negative to positive. Also, F increases where f is positive and F decreases where f is negative.



44. Let $u = \ln x$, $dv = x^{3/2}\,dx \quad \Rightarrow \quad du = \frac{1}{x}\,dx$, $v = \frac{2}{5}x^{5/2}$. Then

$$\begin{aligned}\int x^{3/2}\ln x\,dx &= \tfrac{2}{5}x^{5/2}\ln x - \tfrac{2}{5}\int x^{3/2}\,dx = \tfrac{2}{5}x^{5/2}\ln x - \left(\tfrac{2}{5}\right)^2 x^{5/2} + C \\ &= \tfrac{2}{5}x^{5/2}\ln x - \tfrac{4}{25}x^{5/2} + C\end{aligned}$$

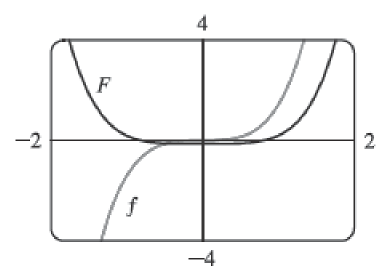
We see from the graph that this is reasonable, since F has a minimum where f changes from negative to positive.



45. Let $u = \frac{1}{2}x^2$, $dv = 2x\sqrt{1+x^2}\,dx \quad \Rightarrow \quad du = x\,dx$, $v = \frac{2}{3}(1+x^2)^{3/2}$.

Then

$$\begin{aligned}\int x^3\sqrt{1+x^2}\,dx &= \tfrac{1}{2}x^2\left[\tfrac{2}{3}(1+x^2)^{3/2}\right] - \tfrac{2}{3}\int x(1+x^2)^{3/2}dx \\ &= \tfrac{1}{3}x^2(1+x^2)^{3/2} - \tfrac{2}{3}\cdot\tfrac{2}{5}\cdot\tfrac{1}{2}(1+x^2)^{5/2} + C \\ &= \tfrac{1}{3}x^2(1+x^2)^{3/2} - \tfrac{2}{15}(1+x^2)^{5/2} + C\end{aligned}$$



We see from the graph that this is reasonable, since F increases where f is positive and F decreases where f is negative. Note also that f is an odd function and F is an even function.

Another method: Use substitution with $u = 1 + x^2$ to get $\frac{1}{5}(1+x^2)^{5/2} - \frac{1}{3}(1+x^2)^{3/2} + C$.

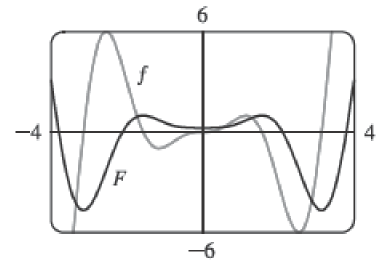
46. First let $u = x^2$, $dv = \sin 2x\,dx \quad\Rightarrow\quad du = 2x\,dx$, $v = -\frac{1}{2}\cos 2x$.

Then $I = \int x^2 \sin 2x\,dx = -\frac{1}{2}x^2\cos 2x + \int x\cos 2x\,dx$.

Next let $U = x$, $dV = \cos 2x\,dx \quad\Rightarrow\quad dU = dx$, $V = \frac{1}{2}\sin 2x$, so

$\int x\cos 2x\,dx = \frac{1}{2}x\sin 2x - \int \frac{1}{2}\sin 2x\,dx = \frac{1}{2}x\sin 2x + \frac{1}{4}\cos 2x + C$.

Thus, $I = -\frac{1}{2}x^2\cos 2x + \frac{1}{2}x\sin 2x + \frac{1}{4}\cos 2x + C$.



We see from the graph that this is reasonable, since F increases where f is positive and F decreases where f is negative. Note also that f is an odd function and F is an even function.

47. (a) Take $n = 2$ in Example 6 to get $\displaystyle\int \sin^2 x\,dx = -\frac{1}{2}\cos x\sin x + \frac{1}{2}\int 1\,dx = \frac{x}{2} - \frac{\sin 2x}{4} + C$.

(b) $\int \sin^4 x\,dx = -\frac{1}{4}\cos x\sin^3 x + \frac{3}{4}\int \sin^2 x\,dx = -\frac{1}{4}\cos x\sin^3 x + \frac{3}{8}x - \frac{3}{16}\sin 2x + C$.

48. (a) Let $u = \cos^{n-1} x$, $dv = \cos x\,dx \quad\Rightarrow\quad du = -(n-1)\cos^{n-2} x\,\sin x\,dx$, $v = \sin x$ in (2):

$$\begin{aligned}\int \cos^n x\,dx &= \cos^{n-1} x\,\sin x + (n-1)\int \cos^{n-2} x\,\sin^2 x\,dx \\ &= \cos^{n-1} x\,\sin x + (n-1)\int \cos^{n-2} x\,(1-\cos^2 x)dx \\ &= \cos^{n-1} x\,\sin x + (n-1)\int \cos^{n-2} x\,dx - (n-1)\int \cos^n x\,dx\end{aligned}$$

Rearranging terms gives $n\int \cos^n x\,dx = \cos^{n-1} x\,\sin x + (n-1)\int \cos^{n-2} x\,dx$ or

$$\int \cos^n x\,dx = \frac{1}{n}\cos^{n-1} x\,\sin x + \frac{n-1}{n}\int \cos^{n-2} x\,dx$$

(b) Take $n = 2$ in part (a) to get $\int \cos^2 x\,dx = \frac{1}{2}\cos x\,\sin x + \frac{1}{2}\int 1\,dx = \dfrac{x}{2} + \dfrac{\sin 2x}{4} + C$.

(c) $\int \cos^4 x\,dx = \frac{1}{4}\cos^3 x\,\sin x + \frac{3}{4}\int \cos^2 x\,dx = \frac{1}{4}\cos^3 x\,\sin x + \frac{3}{8}x + \frac{3}{16}\sin 2x + C$

49. (a) From Example 6, $\int \sin^n x\,dx = -\frac{1}{n}\cos x\ \sin^{n-1} x + \frac{n-1}{n}\int \sin^{n-2} x\,dx$. Using (6),

$$\int_0^{\pi/2} \sin^n x\,dx = \left[-\frac{\cos x\ \sin^{n-1} x}{n}\right]_0^{\pi/2} + \frac{n-1}{n}\int_0^{\pi/2} \sin^{n-2} x\,dx$$

$$= (0-0) + \frac{n-1}{n}\int_0^{\pi/2} \sin^{n-2} x\,dx = \frac{n-1}{n}\int_0^{\pi/2} \sin^{n-2} x\,dx$$

(b) Using $n=3$ in part (a), we have $\int_0^{\pi/2} \sin^3 x\,dx = \frac{2}{3}\int_0^{\pi/2} \sin x\,dx = \left[-\frac{2}{3}\cos x\right]_0^{\pi/2} = \frac{2}{3}$.

Using $n=5$ in part (a), we have $\int_0^{\pi/2} \sin^5 x\,dx = \frac{4}{5}\int_0^{\pi/2} \sin^3 x\,dx = \frac{4}{5}\cdot\frac{2}{3} = \frac{8}{15}$.

(c) The formula holds for $n=1$ (that is, $2n+1=3$) by (b). Assume it holds for some $k \geq 1$. Then

$\int_0^{\pi/2} \sin^{2k+1} x\,dx = \frac{2\cdot 4\cdot 6\cdot\cdots\cdot(2k)}{3\cdot 5\cdot 7\cdot\cdots\cdot(2k+1)}$. By Example 6,

$$\int_0^{\pi/2} \sin^{2k+3} x\,dx = \frac{2k+2}{2k+3}\int_0^{\pi/2} \sin^{2k+1} x\,dx = \frac{2k+2}{2k+3}\cdot\frac{2\cdot 4\cdot 6\cdot\cdots\cdot(2k)}{3\cdot 5\cdot 7\cdot\cdots\cdot(2k+1)}$$

$$= \frac{2\cdot 4\cdot 6\cdot\cdots\cdot(2k)[2\,(k+1)]}{3\cdot 5\cdot 7\cdot\cdots\cdot(2k+1)[2\,(k+1)+1]},$$

so the formula holds for $n=k+1$. By induction, the formula holds for all $n \geq 1$.