

51. Let $u = (\ln x)^n$, $dv = dx \Rightarrow du = n(\ln x)^{n-1}(dx/x)$, $v = x$. By Equation 2,

$$\int (\ln x)^n dx = x(\ln x)^n - \int nx(\ln x)^{n-1}(dx/x) = x(\ln x)^n - n \int (\ln x)^{n-1} dx.$$

52. Let $u = x^n$, $dv = e^x dx \Rightarrow du = nx^{n-1} dx$, $v = e^x$. By Equation 2, $\int x^n e^x dx = x^n e^x - n \int x^{n-1} e^x dx$.

53. $\int \tan^n x dx = \int \tan^{n-2} x \tan^2 x dx = \int \tan^{n-2} x (\sec^2 x - 1) dx = \int \tan^{n-2} x \sec^2 x dx - \int \tan^{n-2} x dx$
 $= I - \int \tan^{n-2} x dx.$

Let $u = \tan^{n-2} x$, $dv = \sec^2 x dx \Rightarrow du = (n-2) \tan^{n-3} x \sec^2 x dx$, $v = \tan x$. Then, by Equation 2,

$$I = \tan^{n-1} x - (n-2) \int \tan^{n-2} x \sec^2 x dx$$

$$1I = \tan^{n-1} x - (n-2)I$$

$$(n-1)I = \tan^{n-1} x$$

$$I = \frac{\tan^{n-1} x}{n-1}$$

Returning to the original integral, $\int \tan^n x dx = \frac{\tan^{n-1} x}{n-1} - \int \tan^{n-2} x dx.$

54. Let $u = \sec^{n-2} x$, $dv = \sec^2 x dx \Rightarrow du = (n-2) \sec^{n-3} x \sec x \tan x dx$, $v = \tan x$. Then, by Equation 2,

$$\int \sec^n x dx = \tan x \sec^{n-2} x - (n-2) \int \sec^{n-2} x \tan^2 x dx$$

$$= \tan x \sec^{n-2} x - (n-2) \int \sec^{n-2} x (\sec^2 x - 1) dx$$

$$= \tan x \sec^{n-2} x - (n-2) \int \sec^n x dx + (n-2) \int \sec^{n-2} x dx$$

so $(n-1) \int \sec^n x dx = \tan x \sec^{n-2} x + (n-2) \int \sec^{n-2} x dx$. If $n-1 \neq 0$, then

$$\int \sec^n x dx = \frac{\tan x \sec^{n-2} x}{n-1} + \frac{n-2}{n-1} \int \sec^{n-2} x dx.$$

55. By repeated applications of the reduction formula in Exercise 51,

$$\int (\ln x)^3 dx = x(\ln x)^3 - 3 \int (\ln x)^2 dx = x(\ln x)^3 - 3[x(\ln x)^2 - 2 \int (\ln x) dx]$$

$$= x(\ln x)^3 - 3x(\ln x)^2 + 6[x(\ln x) - \int (\ln x) dx]$$

$$= x(\ln x)^3 - 3x(\ln x)^2 + 6x \ln x - 6 \int 1 dx = x(\ln x)^3 - 3x(\ln x)^2 + 6x \ln x - 6x + C$$

56. By repeated applications of the reduction formula in Exercise 52,

$$\int x^4 e^x dx = x^4 e^x - 4 \int x^3 e^x dx = x^4 e^x - 4(x^3 e^x - 3 \int x^2 e^x dx)$$

$$= x^4 e^x - 4x^3 e^x + 12(x^2 e^x - 2 \int x e^x dx) = x^4 e^x - 4x^3 e^x + 12x^2 e^x - 24(x e^x - \int x^0 e^x dx)$$

$$= x^4 e^x - 4x^3 e^x + 12x^2 e^x - 24x e^x + 24e^x + C \quad [\text{or } e^x(x^4 - 4x^3 + 12x^2 - 24x + 24) + C]$$

57. The curves $y = x^2 \ln x$ and $y = 4 \ln x$ intersect when $x^2 \ln x = 4 \ln x \Leftrightarrow$

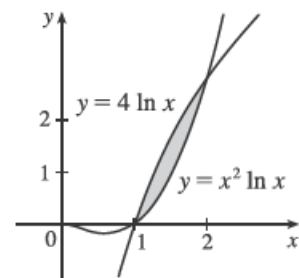
$$x^2 \ln x - 4 \ln x = 0 \Leftrightarrow (x^2 - 4) \ln x = 0 \Leftrightarrow$$

$x = 1$ or 2 [since $x > 0$]. For $1 < x < 2$, $4 \ln x > x^2 \ln x$. Thus,

$$\text{area} = \int_1^2 (4 \ln x - x^2 \ln x) dx = \int_1^2 [(4 - x^2) \ln x] dx. \text{ Let } u = \ln x,$$

$$dv = (4 - x^2) dx \Rightarrow du = \frac{1}{x} dx, v = 4x - \frac{1}{3}x^3. \text{ Then}$$

$$\begin{aligned} \text{area} &= [(\ln x)(4x - \frac{1}{3}x^3)]_1^2 - \int_1^2 \left[(4x - \frac{1}{3}x^3) \frac{1}{x} \right] dx = (\ln 2) \left(\frac{16}{3} \right) - 0 - \int_1^2 (4 - \frac{1}{3}x^2) dx \\ &= \frac{16}{3} \ln 2 - [4x - \frac{1}{9}x^3]_1^2 = \frac{16}{3} \ln 2 - \left(\frac{64}{9} - \frac{35}{9} \right) = \frac{16}{3} \ln 2 - \frac{29}{9} \end{aligned}$$



58. The curves $y = x^2 e^{-x}$ and $y = x e^{-x}$ intersect when $x^2 e^{-x} = x e^{-x} \Leftrightarrow$

$$x^2 - x = 0 \Leftrightarrow x(x - 1) = 0 \Leftrightarrow x = 0 \text{ or } 1.$$

For $0 < x < 1$, $x e^{-x} > x^2 e^{-x}$. Thus,

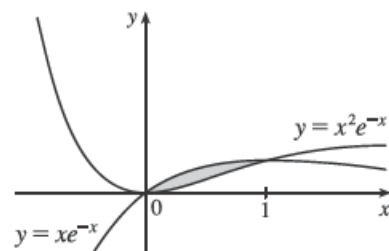
$$\text{area} = \int_0^1 (x e^{-x} - x^2 e^{-x}) dx = \int_0^1 (x - x^2) e^{-x} dx. \text{ Let } u = x - x^2,$$

$$dv = e^{-x} dx \Rightarrow du = (1 - 2x) dx, v = -e^{-x}. \text{ Then}$$

$$\text{area} = [(x - x^2)(-e^{-x})]_0^1 - \int_0^1 [-e^{-x}(1 - 2x)] dx = 0 + \int_0^1 (1 - 2x) e^{-x} dx.$$

Now let $U = 1 - 2x$, $dV = e^{-x} dx \Rightarrow dU = -2 dx$, $V = -e^{-x}$. Now

$$\text{area} = [(1 - 2x)(-e^{-x})]_0^1 - \int_0^1 2e^{-x} dx = e^{-1} + 1 - [-2e^{-x}]_0^1 = e^{-1} + 1 + 2(e^{-1} - 1) = 3e^{-1} - 1.$$



59. The curves $y = \arcsin(\frac{1}{2}x)$ and $y = 2 - x^2$ intersect at

$x = a \approx -1.75119$ and $x = b \approx 1.17210$. From the figure, the area

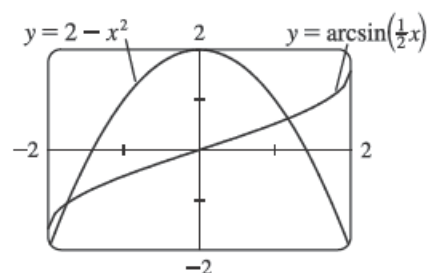
bounded by the curves is given by

$$A = \int_a^b [(2 - x^2) - \arcsin(\frac{1}{2}x)] dx = [2x - \frac{1}{3}x^3]_a^b - \int_a^b \arcsin(\frac{1}{2}x) dx.$$

$$\text{Let } u = \arcsin(\frac{1}{2}x), dv = dx \Rightarrow du = \frac{1}{\sqrt{1 - (\frac{1}{2}x)^2}} \cdot \frac{1}{2} dx, v = x.$$

Then

$$\begin{aligned} A &= \left[2x - \frac{1}{3}x^3 \right]_a^b - \left\{ \left[x \arcsin\left(\frac{1}{2}x\right) \right]_a^b - \int_a^b \frac{x}{2\sqrt{1 - \frac{1}{4}x^2}} dx \right\} \\ &= \left[2x - \frac{1}{3}x^3 - x \arcsin\left(\frac{1}{2}x\right) - 2\sqrt{1 - \frac{1}{4}x^2} \right]_a^b \approx 3.99926 \end{aligned}$$

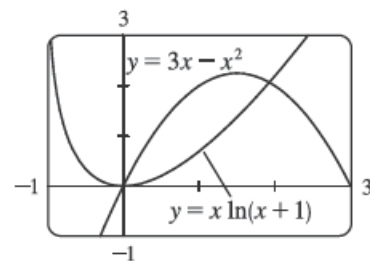


60. The curves $y = x \ln(x+1)$ and $y = 3x - x^2$ intersect at $x = 0$ and $x = a \approx 1.92627$. From the figure, the area bounded by the curves is given by

$$A = \int_0^a [(3x - x^2) - x \ln(x+1)] dx = \left[\frac{3}{2}x^2 - \frac{1}{3}x^3 \right]_0^a - \int_0^a x \ln(x+1) dx.$$

Let $u = \ln(x+1)$, $dv = x dx \Rightarrow du = \frac{1}{x+1} dx$, $v = \frac{1}{2}x^2$. Then

$$\begin{aligned} A &= \left[\frac{3}{2}x^2 - \frac{1}{3}x^3 \right]_0^a - \left\{ \left[\frac{1}{2}x^2 \ln(x+1) \right]_0^a - \frac{1}{2} \int_0^a \frac{x^2}{x+1} dx \right\} \\ &= \left[\frac{3}{2}x^2 - \frac{1}{3}x^3 \right]_0^a - \left[\frac{1}{2}x^2 \ln(x+1) \right]_0^a + \frac{1}{2} \int_0^a \left(x - 1 + \frac{1}{x+1} \right) dx \\ &= \left[\frac{3}{2}x^2 - \frac{1}{3}x^3 - \frac{1}{2}x^2 \ln(x+1) + \frac{1}{4}x^2 - \frac{1}{2}x + \frac{1}{2} \ln|x+1| \right]_0^a \approx 1.69260 \end{aligned}$$



61. $V = \int_0^1 2\pi x \cos(\pi x/2) dx$. Let $u = x$, $dv = \cos(\pi x/2) dx \Rightarrow du = dx$, $v = \frac{2}{\pi} \sin(\pi x/2)$.

$$V = 2\pi \left[\frac{2}{\pi} x \sin\left(\frac{\pi x}{2}\right) \right]_0^1 - 2\pi \cdot \frac{2}{\pi} \int_0^1 \sin\left(\frac{\pi x}{2}\right) dx = 2\pi \left(\frac{2}{\pi} - 0 \right) - 4 \left[-\frac{2}{\pi} \cos\left(\frac{\pi x}{2}\right) \right]_0^1 = 4 + \frac{8}{\pi}(0 - 1) = 4 - \frac{8}{\pi}.$$

62. Volume $= \int_0^1 2\pi x(e^x - e^{-x}) dx = 2\pi \int_0^1 (xe^x - xe^{-x}) dx = 2\pi \left[\int_0^1 xe^x dx - \int_0^1 xe^{-x} dx \right]$ [both integrals by parts]
 $= 2\pi [(xe^x - e^x) - (-xe^{-x} - e^{-x})]_0^1 = 2\pi[2/e - 0] = 4\pi/e$

63. Volume $= \int_{-1}^0 2\pi(1-x)e^{-x} dx$. Let $u = 1-x$, $dv = e^{-x} dx \Rightarrow du = -dx$, $v = -e^{-x}$.

$$V = 2\pi [(1-x)(-e^{-x})]_{-1}^0 - 2\pi \int_{-1}^0 e^{-x} dx = 2\pi [(x-1)(e^{-x}) + e^{-x}]_{-1}^0 = 2\pi [xe^{-x}]_{-1}^0 = 2\pi(0 + e) = 2\pi e$$

64. (a) Use shells about the y -axis:

$$\begin{aligned} V &= \int_1^2 2\pi x \ln x dx \quad \left[\begin{array}{l} u = \ln x, \quad dv = x dx \\ du = \frac{1}{x} dx, \quad v = \frac{1}{2}x^2 \end{array} \right] \\ &= 2\pi \left\{ \left[\frac{1}{2}x^2 \ln x \right]_1^2 - \int_1^2 \frac{1}{2}x dx \right\} = 2\pi \left\{ (2 \ln 2 - 0) - \left[\frac{1}{4}x^2 \right]_1^2 \right\} = 2\pi \left(2 \ln 2 - \frac{3}{4} \right) \end{aligned}$$

- (b) Use disks about the x -axis:

$$\begin{aligned} V &= \int_1^2 \pi (\ln x)^2 dx \quad \left[\begin{array}{l} u = (\ln x)^2, \quad dv = dx \\ du = 2 \ln x \cdot \frac{1}{x} dx, \quad v = x \end{array} \right] \\ &= \pi \left\{ [x(\ln x)^2]_1^2 - \int_1^2 2 \ln x dx \right\} \quad \left[\begin{array}{l} u = \ln x, \quad dv = dx \\ du = \frac{1}{x} dx, \quad v = x \end{array} \right] \\ &= \pi \left\{ 2(\ln 2)^2 - 2 \left([x \ln x]_1^2 - \int_1^2 dx \right) \right\} = \pi \left\{ 2(\ln 2)^2 - 4 \ln 2 + 2[x]_1^2 \right\} \\ &= \pi [2(\ln 2)^2 - 4 \ln 2 + 2] = 2\pi [(\ln 2)^2 - 2 \ln 2 + 1] \end{aligned}$$

$$\begin{aligned}
 65. f_{\text{ave}} &= \frac{1}{b-a} \int_a^b f(x) dx = \frac{1}{\pi/4 - 0} \int_0^{\pi/4} x \sec^2 x dx \quad \left[\begin{array}{l} u = x, \quad dv = \sec^2 x dx \\ du = dx, \quad v = \tan x \end{array} \right] \\
 &= \frac{4}{\pi} \left\{ [x \tan x]_0^{\pi/4} - \int_0^{\pi/4} \tan x dx \right\} = \frac{4}{\pi} \left\{ \frac{\pi}{4} - [\ln |\sec x|]_0^{\pi/4} \right\} = \frac{4}{\pi} \left(\frac{\pi}{4} - \ln \sqrt{2} \right) \\
 &= 1 - \frac{4}{\pi} \ln \sqrt{2} \quad \text{or} \quad 1 - \frac{2}{\pi} \ln 2
 \end{aligned}$$

66.

The rocket will have height $H = \int_0^{60} v(t) dt$ after 60 seconds.

$$\begin{aligned}
 H &= \int_0^{60} \left[-gt - v_e \ln \left(\frac{m - rt}{m} \right) \right] dt = -g \left[\frac{1}{2} t^2 \right]_0^{60} - v_e \left[\int_0^{60} \ln(m - rt) dt - \int_0^{60} \ln m dt \right] \\
 &= -g(1800) + v_e (\ln m)(60) - v_e \int_0^{60} \ln(m - rt) dt
 \end{aligned}$$

Let $u = \ln(m - rt)$, $dv = dt \Rightarrow du = \frac{1}{m - rt}(-r) dt$, $v = t$. Then

$$\begin{aligned}
 \int_0^{60} \ln(m - rt) dt &= \left[t \ln(m - rt) \right]_0^{60} + \int_0^{60} \frac{rt}{m - rt} dt = 60 \ln(m - 60r) + \int_0^{60} \left(-1 + \frac{m}{m - rt} \right) dt \\
 &= 60 \ln(m - 60r) + \left[-t - \frac{m}{r} \ln(m - rt) \right]_0^{60} = 60 \ln(m - 60r) - 60 - \frac{m}{r} \ln(m - 60r) + \frac{m}{r} \ln m
 \end{aligned}$$

So $H = -1800g + 60v_e \ln m - 60v_e \ln(m - 60r) + 60v_e + \frac{m}{r}v_e \ln(m - 60r) - \frac{m}{r}v_e \ln m$. Substituting $g = 9.8$, $m = 30,000$, $r = 160$, and $v_e = 3000$ gives us $H \approx 14,844$ m.

67. Since $v(t) > 0$ for all t , the desired distance is $s(t) = \int_0^t v(w) dw = \int_0^t w^2 e^{-w} dw$.

First let $u = w^2$, $dv = e^{-w} dw \Rightarrow du = 2w dw$, $v = -e^{-w}$. Then $s(t) = [-w^2 e^{-w}]_0^t + 2 \int_0^t w e^{-w} dw$.

Next let $U = w$, $dV = e^{-w} dw \Rightarrow dU = dw$, $V = -e^{-w}$. Then

$$\begin{aligned}
 s(t) &= -t^2 e^{-t} + 2 \left([-w e^{-w}]_0^t + \int_0^t e^{-w} dw \right) = -t^2 e^{-t} + 2 \left(-t e^{-t} + 0 + [-e^{-w}]_0^t \right) \\
 &= -t^2 e^{-t} + 2(-t e^{-t} - e^{-t} + 1) = -t^2 e^{-t} - 2t e^{-t} - 2e^{-t} + 2 = 2 - e^{-t}(t^2 + 2t + 2) \text{ meters}
 \end{aligned}$$

68. Suppose $f(0) = g(0) = 0$ and let $u = f(x)$, $dv = g''(x) dx \Rightarrow du = f'(x) dx$, $v = g'(x)$.

Then $\int_0^a f(x) g''(x) dx = [f(x) g'(x)]_0^a - \int_0^a f'(x) g'(x) dx = f(a) g'(a) - \int_0^a f'(x) g'(x) dx$.

Now let $U = f'(x)$, $dV = g'(x) dx \Rightarrow dU = f''(x) dx$ and $V = g(x)$, so

$\int_0^a f'(x) g'(x) dx = [f'(x) g(x)]_0^a - \int_0^a f''(x) g(x) dx = f'(a) g(a) - \int_0^a f''(x) g(x) dx$.

Combining the two results, we get $\int_0^a f(x) g''(x) dx = f(a) g'(a) - f'(a) g(a) + \int_0^a f''(x) g(x) dx$.

69. For $I = \int_1^4 x f''(x) dx$, let $u = x$, $dv = f''(x) dx \Rightarrow du = dx$, $v = f'(x)$. Then

$$I = [x f'(x)]_1^4 - \int_1^4 f'(x) dx = 4f'(4) - 1 \cdot f'(1) - [f(4) - f(1)] = 4 \cdot 3 - 1 \cdot 5 - (7 - 2) = 12 - 5 - 5 = 2.$$

We used the fact that f'' is continuous to guarantee that I exists.

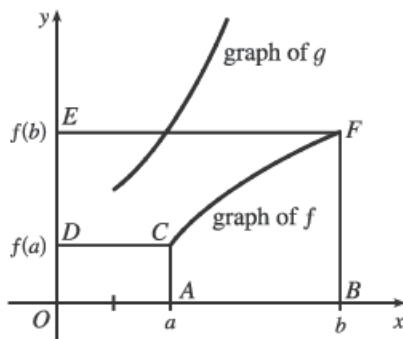
70. (a) Take $g(x) = x$ and $g'(x) = 1$ in Equation 1.

(b) By part (a), $\int_a^b f(x) dx = bf(b) - af(a) - \int_a^b x f'(x) dx$. Now let $y = f(x)$, so that $x = g(y)$ and $dy = f'(x) dx$.

Then $\int_a^b x f'(x) dx = \int_{f(a)}^{f(b)} g(y) dy$. The result follows.

(c) Part (b) says that the area of region $ABFC$ is

$$\begin{aligned} &= bf(b) - af(a) - \int_{f(a)}^{f(b)} g(y) dy \\ &= (\text{area of rectangle } OBF E) - (\text{area of rectangle } OACD) - (\text{area of region } DCFE) \end{aligned}$$



(d) We have $f(x) = \ln x$, so $f^{-1}(x) = e^x$, and since $g = f^{-1}$, we have $g(y) = e^y$. By part (b),

$$\int_1^e \ln x dx = e \ln e - 1 \ln 1 - \int_{\ln 1}^{\ln e} e^y dy = e - \int_0^1 e^y dy = e - [e^y]_0^1 = e - (e - 1) = 1.$$

71. Using the formula for volumes of rotation and the figure, we see that

$$\text{Volume} = \int_0^d \pi b^2 dy - \int_0^c \pi a^2 dy - \int_c^d \pi [g(y)]^2 dy = \pi b^2 d - \pi a^2 c - \int_c^d \pi [g(y)]^2 dy. \text{ Let } y = f(x),$$

which gives $dy = f'(x) dx$ and $g(y) = x$, so that $V = \pi b^2 d - \pi a^2 c - \pi \int_a^b x^2 f'(x) dx$.

Now integrate by parts with $u = x^2$, and $dv = f'(x) dx \Rightarrow du = 2x dx$, $v = f(x)$, and

$$\int_a^b x^2 f'(x) dx = [x^2 f(x)]_a^b - \int_a^b 2x f(x) dx = b^2 f(b) - a^2 f(a) - \int_a^b 2x f(x) dx, \text{ but } f(a) = c \text{ and } f(b) = d \Rightarrow$$

$$V = \pi b^2 d - \pi a^2 c - \pi [b^2 d - a^2 c - \int_a^b 2x f(x) dx] = \int_a^b 2\pi x f(x) dx.$$

72.

(a) We note that for $0 \leq x \leq \frac{\pi}{2}$, $0 \leq \sin x \leq 1$, so $\sin^{2n+2} x \leq \sin^{2n+1} x \leq \sin^{2n} x$. So by the second Comparison Property of the Integral, $I_{2n+2} \leq I_{2n+1} \leq I_{2n}$.

(b) Substituting directly into the result from Exercise 50, we get

$$\frac{I_{2n+2}}{I_{2n}} = \frac{\frac{1 \cdot 3 \cdot 5 \cdots [2(n+1) - 1] \pi}{2 \cdot 4 \cdot 6 \cdots [2(n+1)]} \frac{\pi}{2}}{\frac{1 \cdot 3 \cdot 5 \cdots (2n - 1) \pi}{2 \cdot 4 \cdot 6 \cdots (2n)} \frac{\pi}{2}} = \frac{2(n+1) - 1}{2(n+1)} = \frac{2n+1}{2n+2}$$

(c) We divide the result from part (a) by I_{2n} . The inequalities are preserved since I_{2n} is positive: $\frac{I_{2n+2}}{I_{2n}} \leq \frac{I_{2n+1}}{I_{2n}} \leq \frac{I_{2n}}{I_{2n}}$.

Now from part (b), the left term is equal to $\frac{2n+1}{2n+2}$, so the expression becomes $\frac{2n+1}{2n+2} \leq \frac{I_{2n+1}}{I_{2n}} \leq 1$. Now

$$\lim_{n \rightarrow \infty} \frac{2n+1}{2n+2} = \lim_{n \rightarrow \infty} 1 = 1, \text{ so by the Squeeze Theorem, } \lim_{n \rightarrow \infty} \frac{I_{2n+1}}{I_{2n}} = 1.$$

(d) We substitute the results from Exercises 49 and 50 into the result from part (c):

$$\begin{aligned} 1 &= \lim_{n \rightarrow \infty} \frac{I_{2n+1}}{I_{2n}} = \lim_{n \rightarrow \infty} \frac{\frac{2 \cdot 4 \cdot 6 \cdots (2n)}{3 \cdot 5 \cdot 7 \cdots (2n+1)}}{\frac{1 \cdot 3 \cdot 5 \cdots (2n-1) \pi}{2 \cdot 4 \cdot 6 \cdots (2n)} \frac{\pi}{2}} = \lim_{n \rightarrow \infty} \left[\frac{2 \cdot 4 \cdot 6 \cdots (2n)}{3 \cdot 5 \cdot 7 \cdots (2n+1)} \right] \left[\frac{2 \cdot 4 \cdot 6 \cdots (2n)}{1 \cdot 3 \cdot 5 \cdots (2n-1)} \left(\frac{2}{\pi} \right) \right] \\ &= \lim_{n \rightarrow \infty} \frac{2}{1} \cdot \frac{2}{3} \cdot \frac{4}{3} \cdot \frac{4}{5} \cdot \frac{6}{5} \cdot \frac{6}{7} \cdots \frac{2n}{2n-1} \cdot \frac{2n}{2n+1} \cdot \frac{2}{\pi} \quad [\text{rearrange terms}] \end{aligned}$$

Multiplying both sides by $\frac{\pi}{2}$ gives us the *Wallis product*:

$$\frac{\pi}{2} = \frac{2}{1} \cdot \frac{2}{3} \cdot \frac{4}{3} \cdot \frac{4}{5} \cdot \frac{6}{5} \cdot \frac{6}{7} \cdots$$

(e) The area of the k th rectangle is k . At the $2n$ th step, the area is increased from $2n - 1$ to $2n$ by multiplying the width by

$$\frac{2n}{2n-1}, \text{ and at the } (2n+1)\text{th step, the area is increased from } 2n \text{ to } 2n+1 \text{ by multiplying the height by } \frac{2n+1}{2n}.$$

These two steps multiply the ratio of width to height by $\frac{2n}{2n-1}$ and $\frac{1}{(2n+1)/(2n)} = \frac{2n}{2n+1}$ respectively. So, by part (d), the

$$\text{limiting ratio is } \frac{2}{1} \cdot \frac{2}{3} \cdot \frac{4}{3} \cdot \frac{4}{5} \cdot \frac{6}{5} \cdot \frac{6}{7} \cdots = \frac{\pi}{2}.$$